Precise Charm to Strange Mass Ratio and Light Quark Masses from Full Lattice QCD

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By using a single formalism to handle charm, strange, and light valence quarks in full lattice QCD for the first time, we are able to determine ratios of quark masses to 1%. For m_c/m_s we obtain 11.85(16), an order of magnitude more precise than the current PDG average. Combined with 1% determinations of the charm quark mass now possible this gives $\bar{m}_s(2\text{ GeV}) = 92.4(1.5)$ MeV. The MILC result for $m_s/m_l = 27.2(3)$ yields $\bar{m}_l(2\text{ GeV}) = 3.40(7)$ MeV for the average of u and d quark masses.

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Introduction.—The masses of u, d, and s quarks are some of the least well-known parameters of the standard model. Even the most inaccurate lepton mass (that of the τ) is known to better than 0.01% and yet errors on light quark masses of 30% are quoted in the Particle Data Tables [1]. The reason for the mismatch is the confinement property of the strong force that obscures the connection between the properties of the quark constituents and the hadron physics that is accessible to experiment. To make this connection requires accurate calculations in QCD and accurate experimental results for appropriate hadronic quantities. A method particularly well suited to this is lattice QCD. Here we will demonstrate its use by determining m_c/m_s to 1% and obtaining as a result 1.5% errors for light quark masses, which brings them almost into line with those of heavy quarks.

Heavy quark masses, m_Q, can be determined accurately because $\alpha_s(m_Q)$ is relatively small. 1% errors for charm and bottom quark masses have recently become possible using $\mathcal{O}(\alpha_s^3)$ calculations in QCD perturbation theory for the heavy quark vacuum "bubble" [2] and therefore for the energy-derivative (or time) moments of correlation functions for a heavy quark-antiquark pair at zero momentum. Since the scale of α_s is naturally related to the relevant heavy quark mass, the expressions can be evaluated accurately. To extract the quark mass the perturbative result is compared to a nonperturbative determination containing information from experiment. For a 1^{--} $Q\bar{Q}$ configuration moments of the experimentally measured cross section for ($e^+e^- \to \gamma^* \to$ hadrons) can be used after isolating the heavy quark contribution and using dispersion relations [3]. Alternatively, the time moments for heavy quark current-current correlation functions of various J^{PC} can be directly determined in lattice QCD calculations that have been tuned so that a charmonium or bottomonium mass agrees with experiment [4,5]. The time moments must be extrapolated to the zero lattice spacing (continuum) limit before the comparison to QCD perturbation theory. These two methods give results that agree, with 1% errors for $m_c(3\text{ GeV})$ in the $\overline{\text{MS}}$ scheme. The more traditional "direct" lattice QCD method, although somewhat less accurate, also gives results in good agreement [6]. We can conclude from this that m_c is now accurately known.

The strange quark mass, m_s, being much smaller, cannot be determined this way and is poorly known at present. Instead of a direct determination of m_s, however, we can use the leverage of an accurate result for the ratio m_c/m_s combined with the accurate m_c above [7]. But simple ratios of hadron mass differences give unreliable estimates of m_c/m_s. Two such estimates,

$$\frac{m(B_c) - m(B_u)}{m(B_s) - m(B_u)} = 11, \qquad \frac{m(\Sigma_c) - m(N)}{m(\Sigma) - m(N)} = 6 \qquad (1)$$

differ by almost a factor of 2. The ratio of m_s/m_l [where $m_l = (m_u + m_d)/2$] is known to about 10% from ratios of squared masses of K and π mesons using SU(3) chiral perturbation theory [1]. Clearly neither ratio is determined well enough this way to provide the accuracy we need, because the relationship between hadron mass and well-defined running quark mass is more complicated than these simple ratios must assume.

Lattice QCD, on the other hand, can give very accurate results for the ratio of two quark masses but only if the same formalism is used for both quarks. This has already been used to give accurate results for m_s/m_l, although neither m_s nor m_l is very well determined. Here, for the first time, we give an accurate result for m_c/m_s by using the same formalism for charm, strange and light quarks and this enables us to cascade the accuracy of the heavy quark mass down to the light quarks.

The lattice QCD calculation.—Lattice QCD gives direct access to quark masses through the lattice QCD Lagrangian. Tuning of the masses is done by calculating an appropriate hadron mass and adjusting the quark mass until the hadron mass agrees with experiment. Experimental measurements of appropriate hadron masses are extremely accurate in most cases, with errors at the level of tenths or hundredths of a percent. To make maximum use of this precision we need to calculate the hadron mass in lattice QCD with small statistical and systematic errors. In particular it requires the full effect of sea quarks in the hadron to be included. This is now possible in lattice QCD [8]. Fixing the four quark masses (m_l, m_s, m_c, m_b) from four "gold-plated" hadrons (π, K, η_c, Υ) enables other quantities to be calculated with errors of a few percent and agreement with experiment is obtained [8,9]. This is an important test that QCD, with only one scale parameter and one mass parameter per quark flavor, describes the full range of hadron physics consistently.

The lattice quark mass is a perfectly well-defined running quark mass. However, it is scheme dependent and so varies with the discretization of the Dirac equation used in the lattice calculation. For wider applicability it is more useful to convert the lattice quark mass to a standard continuum scheme such as $\overline{\text{MS}}$. This renormalization has been a major source of systematic error in previous determinations of light and strange quark masses. The best existing result for m_s(2 GeV), with a 7% error, uses the direct method of converting the tuned quark mass in the lattice QCD Lagrangian to the $\overline{\text{MS}}$ scheme using α_s^2 lattice QCD perturbation theory [10]. The error is dominated by the error in the renormalization and it is the error that we will remove here, by instead determining m_c/m_s accurately. The Highly Improved Staggered Quark action [11,12] allows us to use the same discretization of QCD for both charm and strange quarks because it is a fully relativistic "light quark" action that can also be used for charm quarks. Then the mass renormalization factor cancels in the quark mass ratio.

We work with eight different ensembles of gluon field configurations provided by the MILC collaboration. These include the effect of u, d and s sea quarks using the improved staggered quark (asqtad) formalism using the fourth root "trick". This procedure, although "ugly", appears to be a valid discretization of QCD [13–16]. Tests include studies of the Dirac operator and comparisons to effective field theories. Configurations are available with large spatial volumes [$>$2.4 (fm)^3] at multiple values of the light sea masses (using $m_u = m_d = m_l$) and for a wide range of values of the lattice spacing, a. We use configurations at five values of a between 0.15 and 0.05 fm with parameters as listed in Table I.

TABLE I. Ensembles (sets) of MILC configurations used, with size $L^3 \times T$ and sea masses ($\times$ tadpole parameter u_0) m_{0l}^{asq} and m_{0s}^{asq}. The lattice spacing values in units of r_1 after "smoothing" are given in column 2[14]. Column 6 gives the number of configurations and time sources per configuration used for calculating correlators.

Set	r_1/a	$au_0m_{0l}^{\text{asq}}$	$au_0m_{0s}^{\text{asq}}$	L/T	$N_{\text{cf}} \times N_t$
1	2.152(5)	0.0097	0.0484	16/48	631×2
2	2.138(4)	0.0194	0.0484	16/48	631×2
3	2.647(3)	0.005	0.05	24/64	678×2
4	2.618(3)	0.01	0.05	20/64	595×2
5	3.699(3)	0.0062	0.031	28/96	566×4
6	3.712(4)	0.0124	0.031	28/96	265×4
7	5.296(7)	0.0036	0.018	48/144	201×2
8	7.115(20)	0.0028	0.014	64/192	208×2

On these configurations we have calculated quark propagators for charm quarks, strange quarks and light quarks (again $m_u = m_d = m_l$) using the HISQ action. The numerical speed of HISQ means that we have been able to use several nearby quark masses for charm and strange to allow accurate interpolation to the correct values. Table II gives masses for the goldstone pseudoscalar mesons made from either a charm quark-antiquark pair or a strange one (the η_c and the η_s), which are used for tuning. In the charm case, as well as the quark mass, we list the coefficient of the "Naik" term in the HISQ action that corrects for discretization errors through $(am_{0c})^4$. The quark propagators are generated from random wall sources and the goldstone mesons have good signal and noise properties so the meson masses can be determined to high precision using a standard multiexponential fit [17].

The meson masses can be converted to physical units with a determination of the lattice spacing. On an ensemble by ensemble basis this is taken from a parameter in the heavy quark potential called r_1. Values for r_1/a determined by the MILC collaboration [14] are given in Table I. They have errors of 0.3%–0.5%. The physical value for r_1 must then be obtained by comparing to experimentally known quantities and we use the value 0.3133(23) fm obtained from a set of four such quantities, tested for consistency in the continuum limit [18,19].

Using the information about meson masses that we have on each ensemble we can interpolate to the correct ratio for am_{0c} and am_{0s} using appropriate continuum values for the masses of the η_c and η_s. We correct the experimental value of m_{η_c} of 2.9803 GeV to $m_{\eta_c,\text{phys}} = 2.9852(34)$ GeV. This allows for electromagnetic effects (2.4 MeV) [18] and η_c annihilation to gluons (2.5 MeV) [11], both of which are missing from our calculation, so increasing the η_c mass. We take a 50% error on each of these corrections and also increase the experimental error to 3 MeV to allow for the spread of results from different η_c production mechanisms

[1]. Since the total shift is only around 0.2% of the η_c mass it has a negligible effect as can be seen from our error budget below.

The η_s is not a physical particle in the real world because of mixing with other flavor neutral combinations to make the η and η'. However, in lattice QCD, the particle calculated (as here) from only "connected" quark propagtors does not mix and is a well-defined meson. Its mass must be determined by relating its properties to those of mesons such as the π and K that do appear in experiment. From an analysis of the lattice spacing and m_l dependence of the π, K, and η_s masses we conclude that the value of the η_s mass in the continuum and physical m_l limits is 0.6858(40) GeV [18].

The connection between the $\overline{\rm MS}$ mass at a scale μ and the lattice bare quark mass is given by [10,20]

$$\bar{m}(\mu) = \frac{am_0}{a} Z_m(\mu a, m_0 a),$$
$$Z_m = 1 + \alpha_s\left(-\frac{2}{\pi}\log(\mu a) + C + b(am_0)^2 + \ldots\right) + \ldots. \quad (2)$$

From these two equations it is clear that

$$\frac{\bar{m}_c(\mu)}{\bar{m}_s(\mu)} = \frac{am_{0c}}{am_{0s}}\bigg|_{\rm phys}, \quad (3)$$

where phys denotes extrapolation to the continuum limit and physical sea-quark mass limit.

On each ensemble the ratios we have for am_{0c}/am_{0s} then differ from the physical value because of three effects: mistuning from the correct physical meson mass; finite a effects that need to be extrapolated away and effects because the sea light quark masses are not correct. We incorporate these into our fitting function:

$$\frac{m_{0c}}{m_{0s}}\bigg|_{\rm lat} = \frac{m_{0c}}{m_{0s}}\bigg|_{\rm phys}\left(1 + d_{\rm sea}\frac{\delta m_{\rm tot}^{\rm sea}}{m_s}\right) \times\left(1 + \sum_{i,j,k,l} c_{ijkl}\delta_c^i\delta_s^j\left(\frac{am_{\eta_c}}{2}\right)^{2k}(am_{\eta_s})^{2l}\right). \quad (4)$$

$$\delta_c = \frac{m_{\eta_c,\rm MC} - m_{\eta_c,\rm phys}}{m_{\eta_c,\rm phys}}, \qquad \delta_s = \frac{m^2_{\eta_s,\rm MC} - m^2_{\eta_s,\rm phys}}{m^2_{\eta_s,\rm phys}} \quad (5)$$

are the measures of mistuning, where MC denotes lattice values converted to physical units. The last bracket fits the finite lattice spacing effects as a power series in even powers of a. These can either have a scale set by m_c (for which we use $am_{\eta_c}/2$) or by $\Lambda_{\rm QCD}$ (for which we use am_{η_s}). i, j, k, l all start from zero and are varied in the ranges: i, $j \leq 3$, $k \leq 6$, $l \leq 2$ with $i + j + k + l \leq 6$. Doubling any of the upper limits has negligible effect on the final result. The prior on c_{ijkl} is set to 0(1). $\delta m_{\rm tot}^{\rm sea}$ is the total difference between the sea-quark masses used in the simulation and the correct value for $2m_l + m_s$ [18]. This has a tiny effect and we simply use a linear term (adding higher orders has negligible effect). The prior for $d_{\rm sea}$ is 0.0(1). Figure 1 shows the results of the fit, giving m_c/m_s in the continuum limit as 11.85(16) ($\chi^2/{\rm dof} = 0.42$). The error budget is given in Table III.

m_s/m_l is known to 1% from lattice QCD as a byproduct of standard chiral extrapolations of m_π^2 and m_K^2 to the physical point [21]. MILC quote 27.2(3) using asqtad quarks [14]. Our HISQ analysis in [12] gave a result in agreement at 27.8(3), using a Bayesian fit to a function including terms from chiral perturbation theory up to third

TABLE II. Results for the masses in lattice units of the goldstone pseudoscalars made from valence HISQ charm or strange quarks on the different MILC ensembles enumerated in Table I. Columns 2 and 3 give the corresponding bare charm quark mass, and Naik coefficient, respectively. Column 6 gives the bare strange quark mass ($\epsilon = 0$ in that case).

Set	am_{0c}	$1+\epsilon$	am_{η_c}	am_{0s}	am_{η_s}
1	0.81	0.665	2.193 81(16)	0.061	0.504 90(36)
	0.825	0.656	2.220 13(15)	0.066	0.525 24(36)
	0.85	0.641	2.263 52(15)	0.080	0.578 28(34)
2	0.825	0.656	2.219 54(13)	0.066	0.524 58(35)
3	0.65	0.762	1.845 78(8)	0.0537	0.431 18(18)
4	0.63	0.774	1.808 49(11)	0.0492	0.414 36(23)
	0.66	0.756	1.866 74(19)	0.0546	0.436 54(24)
	0.72	0.72	1.981 14(15)	0.054 65	0.436 75(24)
	0.753	0.70	2.042 93(10)	0.06	0.457 87(23)
				0.063	0.469 37(24)
5	0.413	0.893	1.280 57(7)	0.0337	0.294 13(12)
	0.43	0.885	1.316 91(7)	0.0358	0.303 32(12)
	0.44	0.88	1.338 16(7)	0.0366	0.306 75(12)
	0.45	0.875	1.359 34(7)	0.0382	0.313 62(14)
6	0.427	0.885	1.307 31(10)	0.036 35	0.305 13(20)
7	0.273	0.951	0.899 32(12)	0.0228	0.206 21(19)
	0.28	0.949	0.915 51(9)	0.024	0.211 96(13)
8	0.195	0.975	0.671 19(6)	0.0165	0.154 84(14)
				0.018	0.162 09(17)

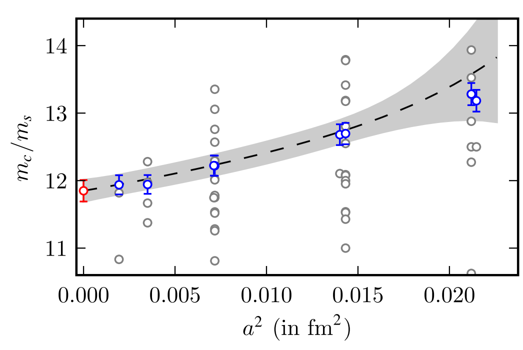

FIG. 1 (color online). Gray points show the raw data for every ratio of m_c/m_s on each ensemble (Table II); these ratios are fit to Eq. (4). The dashed line and associated grey error band (and red point at $a = 0$) show our extrapolation of the resulting tuned m_c/m_s to the continuum limit. Blue points with error bars are from a simple interpolation, separately for each ensemble, to the correct m_c/m_s, and are shown for illustration.

TABLE III. Error budgets for m_c/m_s and m_s/m_l.

	m_c/m_s	m_s/m_l
Overall r_1 uncertainty	0.4%	0.1%
r_1/a uncertainties	0.2	-
Continuum M_{η_c}	0.2	-
Continuum M_{η_s}	1.1	-
Finite volume	-	0.3
a^2 extrapolation, m_q interpolns	0.4	0.8
Sea-quark mass extrapolation	0.0	0.2
Statistical errors	0.3	0.4
Total	1.3%	1.0%

order in m_l and allowing for discretization errors up to and including a^4 and for mixed terms (i.e., m_l-dependent discretization errors). A full error budget is given in Table III; the data are given in [18].

Conclusions.—Our m_c/m_s can be used with any value for m_c to give m_s. The best existing result [4] (converted from $n_f = 4$ to 3) is $\bar{m}_c^{(3)}(2\text{ GeV}) = 1.095(11)$ GeV or $\bar{m}_c^{(3)}(3\text{ GeV}) = 0.990(10)$ GeV. Dividing by 11.85(16) gives $\bar{m}_s^{(3)}(2\text{ GeV}) = 92.4(1.5)$ MeV and $\bar{m}_s^{(3)}(3\text{ GeV}) = 83.5(1.4)$ MeV.

Using the MILC values for m_s/m_l and m_u/m_d (0.42(4) [14]) we can then obtain $\bar{m}_l^{(3)}(2\text{ GeV}) = 3.40(7)$ MeV and $\bar{m}_l^{(3)}(3\text{ GeV}) = 3.07(6)$ MeV; $\bar{m}_u^{(3)}(2\text{ GeV}) = 2.01(14)$ MeV and $\bar{m}_d^{(3)}(2\text{ GeV}) = 4.79(16)$ MeV. The values for all four quark masses are plotted in Fig. 2 in comparison to the current evaluations from the Particle Data Tables [1].

Thus our high accuracy on m_c/m_s allows us to leverage 2% accurate values for m_s and m_l that are completely nonperturbative in lattice QCD, for the first time. Our m_s mass is higher, by around 1σ, than our previous value of $\bar{m}_s(2\text{ GeV}) = 87(6)$ MeV which used 2-loop lattice QCD perturbation theory [10]. Then the error was dominated by unknown α_s^3 terms. Our new result, which does not have this limitation, has an error almost 5 times smaller. Our new error is almost an order of magnitude smaller than other lattice QCD results from full QCD [22,23]. These use direct methods of converting the lattice mass to the $\overline{\text{MS}}$ mass, and have 10% errors.

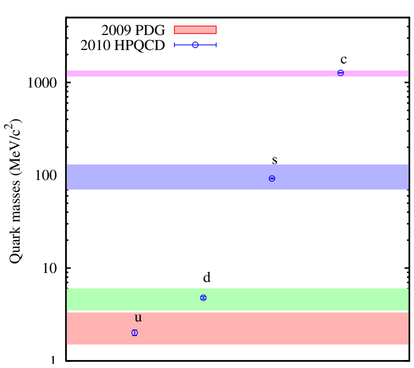

FIG. 2 (color online). Our results for the 4 lightest quark masses compared to the current PDG evaluations (shaded bands) [1]. Each mass is quoted in the $\overline{\text{MS}}$ scheme at its conventional scale: 2 GeV for u, d, s ($n_f = 3$); m_c for c ($n_f = 4$).

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[19] Note that [4] determined m_c using an earlier and less accurate value of r_1 [0.321(5) fm]. We have checked that using the new value does not change m_c significantly.
[20] The mass parameter am_{0c} in Eq. (3) should really be the perturbative pole mass at tree level [11], which differs from am_{0c} by $\approx 0.04(am_{0c})^4$, but gives the same extrapolated m_c/m_s ratio.
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