Newton's Cradle versus Nonbinary Collisions

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Newton's cradle is a classical example of a one-dimensional impact problem. In the early 1980s the naive perception of its behavior was corrected: For example, the impact of a particle does *not* exactly cause the release of the farthest particle of the target particle train, if the target particles have been just in contact with their own neighbors. It is also known that the naive picture would be correct if the whole process consisted of purely binary collisions. Our systematic study of particle systems with truncated power-law repulsive force shows that the quasibinary collision is recovered in the limit of hard core repulsion, or a very large exponent. In contrast, a discontinuous steplike repulsive force mimicking a hard contact, or a very small exponent, leads to a completely different process: the impacting cluster and the targeted cluster act, respectively, as if they were nondeformable blocks.

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Impacts of more than two bodies in one dimension arouse many interesting phenomena. A classical example is Newton's cradle: Before the collision, n_1 identical particles are just in contact with their neighbor(s) and moving together with a velocity v_0, while other n_2 particles being also identical to the former ones are just in contact with their neighbor(s) and are staying at rest ahead of the aforementioned n_1 particles. We shall denote this setup the $(n_1 \triangleright n_2)$ collision. Figure 1 shows the case of $(1 \triangleright n-1)$ collision.

Except for the $(1 \triangleright 1)$ collision, i.e., binary collision, the conservation laws of energy and momentum are not sufficient to determine the final velocities of all particles [1]. Nevertheless, outside the physicists community, it has been widely believed that the $(n_1 \triangleright n_2)$ collision ends up with ejecting the farthest n_1 particles with the initial velocity v_0. In fact, it is not exactly what occurs [1–4]. Using a high-speed camera, Donahue *et al.* [5] observed directly that the $(1 \triangleright 2)$ collision of steel balls causes the bouncing back of the impacting particle. Experimentally, the simple mass and spring model has been shown to describe well the observations [5–7] if the springs obey the Hertz force law [8], i.e., repulsive force with $3/2$th power of the positive "overlap" between the particles and zero otherwise. Theoretically, the case of Hertz-type force as well as that of the truncated harmonic force (i.e., Hooke's law for the positive overlap and zero otherwise) has been studied in detail; the relation with the dispersion law was discussed [1,2], and the propagation of solitary waves has also been studied both theoretically and experimentally [3,6,7,9,10]. Numerical and analytical approximation studies also explored the behaviors of large n_2 cases [11,12].

While these studies have elucidated the phenomenon of multibody collisions, they focused rather on particular force laws, such as of the Hertz force or of truncated harmonic force; the systematic study about different force laws is still lacking to the author's knowledge (cf. [11]). Such study may shed light on these realistic cases from a different point of view, and may also give an insight about the contact between hard spheres, which involves some nondeterminism [13]. This nondeterminism is related to the notion of hard sphere: a discontinuous infinite potential barrier at a critical distance requires a proper definition as a limiting procedure. One way would be to use a power-law repulsive potential and let its exponent go to infinity ($\alpha \to \infty$, see below). Hereafter we shall call this limit the hard core limit or hard core repulsion. The other way would be to use a linear potential slope and let its gradient go to infinity ($\alpha \to 0$, see below). We shall call this limit the steplike force. We will show that these alternatives are not equivalent with each other in the context of the contact between hard spheres and, therefore, in the study of $(2 \triangleright 3)$ collision, for example. While a simple picture of the network of quasibinary collisions for Newton's cradle is compatible with the hard core repulsion [2,14], the model with the steplike force leads to completely different and somehow complementary phenomenon.

We are, therefore, motivated to study systematically different force laws, $F_\alpha(\delta)$, between the two neighboring particles:

$$F_\alpha(\delta) = \begin{cases} a\delta^\alpha & (\delta \geq 0) \\ 0 & (\delta < 0), \end{cases} \tag{1}$$

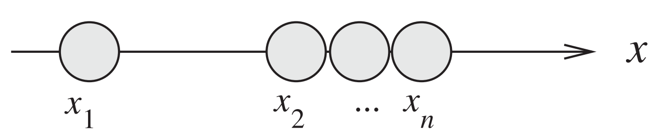

FIG. 1. Setup of $(1 \triangleright n-1)$ collision (see the text). Before the collision, all the latter $(n-1)$ particles are at rest and just in contact with its neighbor(s).

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where a is a positive constant and the overlap δ is the distance approached with respect to the point of "just in contact" ($\delta = 0$). Depending on the value of α, the force represents, for example,

$$\begin{aligned} &\alpha \gg 1 \quad \text{(hard core repulsion)}, \\ &\alpha = \frac{3}{2} \quad \text{(Hertz contact force)} \\ &\alpha = 1 \quad \text{(truncated harmonic force)} \\ &\alpha \ll 1 \quad \text{(steplike force)}. \end{aligned} \tag{2}$$

The truncated harmonic case is analytically solvable and serves to check the numerical calculations. (We ignore the effect of the internal degrees of freedom of each particle [15,16] or adhesive interaction among particles [17], in the light of fairly good reproduction of experimental results by the Hertz force [5–7].) As mentioned above, the hard core limit corresponds to $\alpha \gg 1$. In the opposite limit, the steplike force ($\alpha \ll 1$), the repulsive force is discontinuously switched across the contact point ($\delta = 0$).

Our system is defined as follows. We denote by x_i and p_i ($i = 1, \ldots, n$) the position and momentum of the ith movable particle of a common mass M. We assume the spatial ordering, $x_1 \leq x_2 \leq \cdots \leq x_n$. The "radius" of the particles R is also assumed to be common. Then the overlap δ between the neighboring particle pair ($i, i+1$) is $\delta = 2R - x_i + x_{i+1}$. With a given force law (1) Newton's equations of motion are identical to those studied before [2–7]:

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{p_i}{M}, \\ \frac{dp_i}{dt} &= F_\alpha(2R - x_i + x_{i-1}) - F_\alpha(2R - x_{i+1} + x_i), \end{aligned} \tag{3}$$

for $i = 1, \ldots, n$. We define $x_0 = -\infty$ and $x_{n+1} = +\infty$ with $n = n_1 + n_2$ in the setup of ($n_1 \triangleright n_2$) collision (Fig. 1). In this setup a simple dimensional analysis tells that the values of the mass M, force parameter a, injecting velocity v_0, and the radius R can be set equal to unity without losing generality [3]. (In particular, R can be eliminated by using the displacements $u_i(t) = x_i(t) - 2Ri +$ const with respect to the positions at the moment of initial impact. In other words, the ratio of the ultimate velocities v_i of the ith particle after the whole impact process to the impacting velocity v_0 is a function of the exponent α and the pair (n_1, n_2) alone. Hereafter, we therefore use the units $M = a = v_0 = R = 1$.

Figure 2 shows the ($1\triangleright 2$) collisions (left) and ($2\triangleright 3$) collisions (right), with the force exponents $\alpha = 10$ (top), $3/2$ (middle), and $1/10$ (bottom), respectively. We integrated Eq. (3) using MATHEMATICA$^{\text{TM}}$. The errors in the total momentum (=1) and the total energy (1/2) are, respectively, of order of 10^{-12} and 10^{-6}.

On the top row in Fig. 2, the force with $\alpha = 10$ mimics the hard core repulsion. The outcomes are what we expect

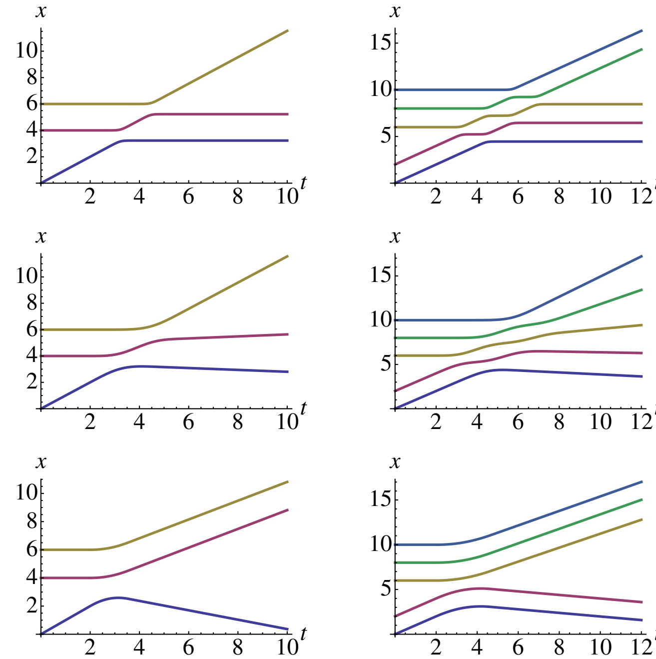

FIG. 2 (color online). Time courses of the ($1\triangleright 2$) collisions (left column) and the ($2\triangleright 3$) collisions (right column) on the x vs t plane. The exponent α of the truncated repulsive force (1) is 10 (top row), 3/2 (middle row), and 1/10 (bottom row).

from the naive picture of Newton's cradle. At the same time, it is clearly visible that the collision consists of a network of (quasi)binary collisions [2,14]. Such a simple explanation may have been given somewhere, but we did not find it. In this case the particles behave as if they had enough spaces between their neighbors although the target particles were just in contact at the moment of collision. Such a somewhat paradoxical result is ascribed to the hard core repulsion, which rises steeply only at the overlap $\delta \sim 1$ but remains very soft for $0 \leq \delta < 1$. In fact, the results for $\alpha = 10$ are robust against the addition of small (true) gaps between the particles before the impact.

For the Hertz repulsive interaction (middle row in Fig. 2), the impacting particles [$x_1(t)$ (left) or $x_1(t)$ and $x_2(t)$ (right)] undergo the bouncing back, as reported previously [3–7]. This is a generic result for any finite positive values of α (see below). For $\alpha = 1$ we can compare the numerical result with the analytic result of, e.g., the ($1\triangleright 2$) collision. Mathematically, for any ($1\triangleright n-1$) collision with $n \leq 3$, the ultimate momentum p_1 *must* be negative unless all the momentum is transmitted to the farthest particle p_n [18].

Finally, the force with $\alpha = 1/10$ (bottom row in Fig. 2) mimics the steplike force; the force raises abruptly upon the overlapping, $\delta > 0$. It is a surprise that the groups of particles, impacting group on the one hand and target group on the other hand, keep their identity and move as if they were nondeformable blocks. Except for the interface between these two groups, the overlap between the neighboring particles is kept almost zero throughout the process. [For the (1$\triangleright$2) collision, δ for the pair (x_2, x_3) is 4×10^{-4} at the maximum, while for the (2$\triangleright$3) collision, δ for the pairs (x_1, x_2), (x_3, x_4), and (x_4, x_5) are up to 10^{-3}, 1.5×10^{-2}, and 1.5×10^{-5}, respectively.] We will give later a physical argument for such nondeformable blocks in the limit of $\alpha \to 0$.

In order to see systematically the momentum redistribution through the (1$\triangleright$2) collision, we plotted the ultimate momenta $(p_1,\ p_2,\ p_3)$ on the plane of the conserved momentum, $p_1 + p_2 + p_3 = 1$. Figure 3 views this plane perpendicularly, i.e., from the direction $p_1 = p_2 = p_3$. On this plane the energy conservation, $p_1^2 + p_2^2 + p_3^2 = 1$, defines the circle that passes through the three vertices $(p_1, p_2, p_3) = (1, 0, 0)$ (the initial condition, left bottom corner), $(0, 1, 0)$ (right bottom corner), and $(0, 0, 1)$ (top corner) of the regular triangle. For each value of α the ultimate momenta are represented as a thick dot. As α increases, the ultimate momenta approach the vertex $(0, 0, 1)$ (cf. top-left figure of Fig. 2). By contrast, in the limit of small α the momenta looks to converge at the midpoint of the arc, $(-1/3,\ 2/3,\ 2/3)$ (cf. bottom-left figure of Fig. 2).

How can the steplike force among particles ($\alpha \to 0$) lead to the behavior of nondeformable blocks? A semiquantitative explanation is as follows. For $\alpha \to 0$ the truncated force is the Heaviside step function, $F_{0+}(\delta) = \theta(\delta)$. Let us take a (1$\triangleright n-1$) collision, as an example. As soon as the impacting particle (x_1) has an overlap $\delta > 0$ with the closest target particle (x_2), the force accelerates the latter particle and leads to the overlap between this particle and the next particle x_3, and so on. Shortly after the first overlap, the second equation of (3) reads, therefore, $dp_1/dt = -1$, $dp_2/dt = \cdots = dp_{n-1}/dt = 0$, and $dp_n/dt = 1$. Since the propagation of the overlap takes little time for small $\alpha(>0)$, the overlaps between the neighboring particles are all small. In this stage, the farthest particle (x_n) being accelerated should soon detach from the $(n-1)$th particle. It is then the latter particle x_{n-1} that is accelerated, and detaches from x_{n-2}, and so on. The events of acceleration and gap creation will, therefore, backpropagate up to the pair (x_2, x_3). Now the story will restart again, and it will continue until the impacting particle x_1 definitively leaves its neighbor x_2. Throughout the process, all the neighboring particle pairs except for the interface pair (x_1, x_2) remain almost just in contact. Despite such a pathological process [19], we can calculate the ultimate momenta of the nondeformable blocks, merely by assuming the nondeformability, i.e., $\delta = 0$ for all neighboring pairs except at the interface of the clusters: After some time-coarse graining of the momenta, $p_2, \ldots, p_n$, which we denote with the over bar, the equations of motion *during* the collision become

$$\bar{p}_2 = \cdots = \bar{p}_n \equiv p', \tag{4}$$

and

$$\frac{dp_1}{dt} = -1, \qquad (n-1)\frac{dp'}{dt} = 1. \tag{5}$$

The problem is thus reduced to the binary collision between the masses 1 and $(n-1)$. The ultimate momenta are then given as $p_1 = -(n-2)/n$ and $p' = 2/n$ just by the laws of conservation. For the ($n_1 \triangleright n_2$) collision, the same argument leads to $p_1 = \cdots = p_{n_1} = (n_1 - n_2)/(n_1 + n_2)$ and $p_{n_1+1} = \cdots = p_{n_1+n_2} = 2n_2/(n_1 + n_2)$. Our numerical results with $\alpha = 1/10$ approximately reproduces the above formula of ultimate momenta for (1$\triangleright$2), (1$\triangleright$3), (1$\triangleright$4), (2$\triangleright$2), and (2$\triangleright$3) collisions within 1% of deviations. We should note that the system with steplike force is not robust against the insertion of gaps between the neighboring particle pairs. For $\alpha = 1/10$ the initial interparticle gap of 0.1 within each cluster is sufficient to render all processes as a network of binary collisions. For general α, a dimensionless number characterizing the importance of a gap ϵ for a (1$\triangleright n-1$) collision is the ratio between the injected kinetic energy $p_0^2/(2M)$ and the characteristic potential energy $a\epsilon^{\alpha+1}/(\alpha+1)$. If a is very large, the injected momentum p_0 should be very large in order for the nonbinary collisions to come into play. This explains the observation [5] that the smallest gaps between the particles before the collision eliminated the rebounding of the impacting particle.

It is among future problems to study inhomogeneous systems. A typical example is the system with a hard wall, which transmits the momentum but not the energy. In [20] the authors studied experimentally the impact of n hard spheres in contact and moving at the same velocity v_0 against a wall at rest. They observed that, when n is large enough, the furthermost particles from the wall bounce successively whereas the last 5 particles nearest to the wall bounce in block. For very large n and very small α, the numerical calculation is very delicate and efficient

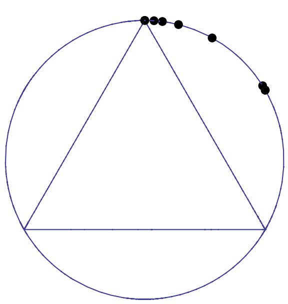

FIG. 3 (color online). Ultimate momenta of three particles (thick dots) with different values of the force exponent, $\alpha =$ 10, 5, 2, 3/2, 1, 1/2, 1/5, and 1/10 in increasing order of the distances from the top vertex of the triangle. See the text for the reading out of the momenta. The first two points for $\alpha = 10$ and 5 are not distinguishable in the figure, while the last two points ($\alpha = 1/5$ and 1/10) are very close with each other and to the midpoint, $(-1/3, 2/3, 2/3)$.

schemes are under exploration [21]. Instead, we did preliminary studies of the (2▷2) collisions in the presence of a rigid wall just in contact with the farthest target particle (x_4). For $\alpha = 10$ the collisions ended up with bouncing back of the two injected particles just in contact, as expected by the picture of quasibinary collision. For $\alpha = 1/10$, however, all four particles left the wall at comparable velocities (data not shown). The wall or inhomogeneity, therefore, prevented the target particles from behaving as a nondeformable cluster.

In conclusion, our systematic study about different laws of interparticle force clarified several things: The notions of contact between hard spheres became clearer, and we understood qualitatively the prevailing naive notion of Newton's cradle in terms of a network of quasibinary collisions. The solitary wave and its propagation studied for the Hertz force [3,6,7,9] can be regarded as an interpolation to $\alpha < \infty$ of the above-mentioned quasibinary collisions, as suggested by [3]. The hitherto unexplored case of steplike force leads to a complementary feature of collisions, where the group of impacting particles and that of target particles behave, respectively, as nondeformable blocks. It will be noteworthy that, in continuum approximation, an estimation [11] suggested the divergence of the width of their mechanical pulse for $\alpha \to 0$ (cf. [22]) although the authors of [11] have not considered the case with $\alpha < 1$. Are interactions with $\alpha < 1$ realizable? The steplike force ($\alpha = 0$) is reminiscent of the interface energy of two-fluid interface [23]. Also a uniform long-range force (gravity, electrostatic force between plates, etc.) could be devised to work in a macroscopic setup. Unlike Hertz force, the restoring force of rubber balloon obeys $\alpha = 1$ for spherical balloons (data not shown) and should obey $\alpha = 1/2$ for cylindrical balloons contacting side by side. The impact problem of soft materials may show different aspects from the hard ones. For example, simple dimensional analysis shows that, for $\alpha < 1$, the negligence of the rag time due to intraparticle elastic wave becomes a better approximation for small impact velocities while the opposite is the case for $\alpha > 1$. Even outside pure scientific interests such as granular materials and soft matters, the redistribution of injected momentum may play important roles in both macroscopic and microscopic phenomena, e.g., in the martial arts [24], biomechanics [25], robotics [26], or composite materials [27].

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[1] F. Herrmann and P. Schmälzle, Am. J. Phys. **49**, 761 (1981).
[2] F. Herrmann and M. Seitz, Am. J. Phys. **50**, 977 (1982).
[3] E. J. Hinch and S. Saint-Jean, Proc. R. Soc. A **455**, 3201 (1999).
[4] S. Hutzler, G. Delaney, D. Weaire, and F. MacLeod, Am. J. Phys. **72**, 1508 (2004).
[5] C. M. Donahue, C. M. Hrenya, A. P. Zelinskaya, and K. J. Nakagawa, Phys. Fluids **20**, 113301 (2008).
[6] C. Coste, E. Falcon, and S. Fauve, Phys. Rev. E **56**, 6104 (1997).
[7] C. Coste and B. Gilles, Eur. Phys. J. B **7**, 155 (1999).
[8] K. L. Johnson, *Contact Mechanics* (Cambridge University Press, Cambridge, England, 1987).
[9] V. F. Nesterenko, J. Appl. Mech. Tech. Phys. **24**, 733 (1983).
[10] G. Friesecke and K. Matthies, Physica (Amsterdam) **171D**, 211 (2002).
[11] A. Rosas and K. Lindenberg, Phys. Rev. E **69**, 037601 (2004).
[12] U. Harbola, A. Rosas, M. Esposito, and K. Lindenberg, Phys. Rev. E **80**, 031303 (2009).
[13] V. Ceanga and Y. Hurmuzlu, J. Appl. Mech. **68**, 575 (2001).
[14] D. T. Wu, Physica (Amsterdam) **315A**, 194 (2002).
[15] S. I. Nagahiro and Y. Hayakawa, Phys. Rev. E **67**, 036609 (2003).
[16] Y. Tanaka, Phys. Rev. E **73**, 031403 (2006).
[17] N. V. Brilliantov, N. Albers, F. Spahn, and T. Poschel, Phys. Rev. E **76**, 051302 (2007).
[18] The momentum constraint, $\sum_{i=1}^{n} p_i = 1$, and the conditions $p_i \geq 0$ for all i define a $(n-1)$-dimensional simplex. With this simplex, $(n-1)$ sphere defined by the energy constraint, $\sum_{i=1}^{n} p_i^2 = 1$, shears only the corner points of the former.
[19] For small but finite values of α, smoothed trajectories replace the pathological process.
[20] E. Falcon, C. Laroche, S. Fauve, and C. Coste, Eur. Phys. J. B **5**, 111 (1998).
[21] Physical Numerics Collaboration (to be published).
[22] V. Nesterenko, in *Symposium BB: The Granular State*, edited by S. Sen and M. Hunt, MRS Symposia Proceedings No. 627 (Materials Research Society, Pittsburgh, 2001), paper BB3.1.
[23] M. Schindler (private communication).
[24] M. Sprague, *Fighting Science: The Laws of Physics for Martial Artists* (Turtle Press, Santa Fe, NM, 2002).
[25] M. Alam, M. Wahab, and C. Jenkins, Mech. Mater. **39**, 145 (2007).
[26] T. Umedachi, T. Kitamura, K. Takeda, T. Nakagaki, R. Kobayashi, and A. Ishiguro, in *Distributed Autonomous Robotic Systems 8* (Springer, Berlin, 2009), pp. 193–202.
[27] S. Abrate, Compos. Struct. **51**, 129 (2001).