Gluon Thermodynamics at Intermediate Coupling

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We calculate the thermodynamic functions of Yang-Mills theory to three-loop order using the hard-thermal-loop perturbation theory reorganization of finite temperature quantum field theory. We show that at three-loop order hard-thermal-loop perturbation theory is compatible with lattice results for the pressure, energy density, and entropy down to temperatures $T \sim 2$–$3T_c$.

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The goal of ultrarelativistic heavy-ion collision experiments is to generate energy densities and temperatures high enough to create a plasma of quarks and gluons called the quark-gluon plasma. One of the chief theoretical questions which has emerged in this area is whether it is more appropriate to describe the state of matter generated during these collisions using weakly coupled quantum field theory or a strong-coupling formalism based on AdS/CFT correspondence. Early data from the Relativistic Heavy-Ion Collider (RHIC) at Brookhaven National Labs indicated that the state of matter created there behaved more like a fluid than a plasma and that this "quark-gluon fluid" was strongly coupled [1].

In the intervening years, however, work on the perturbative side has shown that observables like jet quenching [2] and elliptic flow [3] can also be described using a perturbative formalism. Since in phenomenological applications predictions are complicated by the modeling required to describe, for example, initial state effects, the space-time evolution of the plasma, and hadronization of the plasma, there are significant theoretical uncertainties remaining. Therefore, one is hard put to conclude whether the plasma is strongly or weakly coupled based solely on RHIC data. To have a cleaner testing ground one can compare theoretical calculations with results from lattice quantum chromodynamics (QCD).

Looking forward to the upcoming heavy-ion experiments scheduled to take place at the Large Hadron Collider (LHC) at the European Organization for Nuclear Research (CERN) it is important to know if, at the higher temperatures generated, one expects a strongly coupled (liquid) or weakly coupled (plasma) description to be more appropriate. At RHIC, initial temperatures on the order of 1 to 2 times the QCD critical temperature, $T_c \sim 190$ MeV, were generated. At LHC, initial temperatures on the order of 4–$5T_c$ are expected. The key question is, will the generated matter behave more like a plasma of quasiparticles at these higher temperatures?

In this Letter we discuss the calculation of thermodynamic functions of a gas of gluons at phenomenologically relevant temperatures. We present results at leading order (LO), next-to-leading order (NLO), and next-to-next-to-leading order (NNLO) and compare with available lattice data [4] for the thermodynamic functions of SU(3) Yang-Mills theory. The calculation is based on a reorganization of the theory around hard-thermal-loop (HTL) quasiparticles. Our results indicate that the lattice data at temperatures $T \gtrsim 2$–$3T_c$ are consistent with the quasiparticle picture. This is a nontrivial result since, in this temperature regime, the QCD coupling constant is neither infinitesimally weak nor infinitely strong with $g_s \sim 2$, or equivalently $\alpha_s = g_s^2/(4\pi) \sim 0.3$. Therefore, we have a crucial test of the quasiparticle picture in the intermediate coupling regime.

The calculation of thermodynamic functions using weakly coupled quantum field theory has a long history [5]. The QCD free energy is known up to order $g_s^6 \log(g_s)$; however, the resulting weak-coupling approximations do not converge at phenomenologically relevant couplings. For example, simply comparing the magnitude of low-order contributions to the QCD free energy with three quark flavors ($N_f = 3$), one finds that the g_s^3 contribution is smaller than the g_s^2 contribution only for $g_s \lesssim 0.9$ ($\alpha_s \lesssim 0.07$) which corresponds to a temperature of $T \sim 10^5$ GeV $\sim 5 \times 10^5 T_c$.

The poor convergence of finite-temperature perturbative expansions of thermodynamic functions stems from the fact that at high temperature the classical solution is not described by massless gluonic states. Instead one must include plasma effects such as the screening of electric fields and Landau damping via a self-consistent hard-thermal-loop resummation. There are several ways of systematically reorganizing the perturbative expansion [6]. Here we will present a new NNLO calculation which uses the hard-thermal-loop perturbation theory (HTLpt) method [7–9] and compare with previously obtained LO and NLO results.

The basic idea of the technique is to add and subtract an effective mass term from the bare Lagrangian and to associate the added piece with the free part of the

Lagrangian and the subtracted piece with the interactions [10,11]. However, in gauge theories, one cannot simply add and subtract a local mass term since this would violate gauge invariance. Instead, one adds and subtracts an HTL improvement term which modifies the propagators and vertices self-consistently so that the reorganization is manifestly gauge invariant [12].

***Formalism*.**—The Lagrangian density for SU(N_c) Yang-Mills theory in Minkowski space is

$$\mathcal{L}_{\rm YM} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_{\rm gf} + \mathcal{L}_{\rm gh} + \Delta\mathcal{L}_{\rm YM}. \tag{1}$$

Here the field strength is $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + ig_s[A^\mu, A^\nu]$, with A^μ an element of the SU(N_c) gauge group. The ghost term $\mathcal{L}_{\rm gh}$ depends on the gauge-fixing term $\mathcal{L}_{\rm gf}$. We use $\overline{\rm MS}$ dimensional regularization with a renormalization scale μ and covariant gauge-fixing $\mathcal{L}_{\rm gf} = -(\partial_\mu A^\mu)^2/(2\xi)$ where ξ is the gauge parameter. HTLpt is gauge-fixing independent; therefore, all results shown below are independent of the gauge-fixing parameter. The independence of the results on the gauge parameter was explicitly demonstrated in general covariant and Coulomb gauges in Ref. [8].

HTLpt is a reorganization of the perturbation series for thermal gauge theories. In the case of Yang-Mills theory, the Lagrangian density is written as

$$\mathcal{L} = (\mathcal{L}_{\rm YM} + \mathcal{L}_{\rm HTL})|_{g_s\to\sqrt{\delta}g_s} + \Delta\mathcal{L}_{\rm HTL}, \tag{2}$$

where $\Delta\mathcal{L}_{\rm HTL}$ collects counterterms necessary to account for additional divergences introduced by $\mathcal{L}_{\rm HTL}$. The HTL improvement term is

$$\mathcal{L}_{\rm HTL} = -\frac{1}{2}(1-\delta)m_D^2\,{\rm Tr}\left(F_{\mu\alpha}\left\langle\frac{y^\alpha y^\beta}{(y\cdot D)^2}\right\rangle_y F^\mu{}_\beta\right), \tag{3}$$

where $D^\mu = \partial^\mu + ig_sA^\mu$ is the covariant derivative, $y^\mu = (1, \hat{\mathbf{y}})$ is a lightlike four vector, and $\langle\ldots\rangle_y$ represents an average over the directions of $\hat{\mathbf{y}}$. The term (3) has the form of the effective Lagrangian that would be induced by a rotationally invariant ensemble of color-charged sources in the eikonal approximation. The free parameter m_D can be identified with the Debye screening mass, but is not assumed to be $m_D \sim g_sT$ at leading order. HTLpt is defined by treating δ as a formal expansion parameter and expanding in a power series in δ around $\delta = 0$. This generates loops with fully dressed propagators and vertices and also automatically generates the counterterms necessary to remove the dressing as one proceeds to higher loop orders [7–9].

If the expansion in δ could be calculated to all orders, the final result would not depend on m_D. However, any truncation of the expansion in δ produces results that depend on m_D. We will first obtain the thermodynamic potential $\Omega(T, \alpha_s, m_D, \mu, \delta = 1)$ which is a function of the mass parameter m_D. A prescription is then required to determine m_D as a function of T and α_s. The canonical way to fix the Debye mass in HTLpt is to require the thermodynamic potential to satisfy a variational equation; however, at NNLO this results in a complex-valued Debye mass which causes the thermodynamic functions to also become complex. The same problem occurs in QED [9] and scalar theories [10] and in the case of gauge theories is most likely due to the expansion we perform of the resulting integrals in m_D/T (see next section). Because of the complexity of the variational Debye mass, here we will use a NLO perturbative mass prescription detailed below. This prescription guarantees that the Debye mass and hence thermodynamic functions are real valued at all temperatures.

After having fixed the Debye mass as a function of T and α_s, the free energy $\mathcal{F}$ is obtained by evaluating the thermodynamic potential at the appropriate value of the Debye mass. The pressure, energy density, and entropy are then evaluated using standard thermodynamic relations $\mathcal{P} = -\mathcal{F}$, $\mathcal{E} = \mathcal{F} - T\frac{d\mathcal{F}}{dT}$, $\mathcal{S} = -\frac{d\mathcal{F}}{dT}$.

***Thermodynamic potentials*.**—In this section we present the final renormalized thermodynamic potential at orders δ^0 (LO), δ^1 (NLO), and δ^2 (NNLO). The LO and NLO results were first obtained in [8] and we list them here for completeness. The thermodynamic potentials are computed using a dual expansion in g_s and m_D which assumes that at leading order m_D/T is $\mathcal{O}(g_s)$. We then only include terms which contribute naively through order g_s^5 [8]. This dual truncation is not necessary in principle; however, in practice it makes the calculation tractable. We note that the expansion in m_D/T does not spoil the gauge invariance of our final results since the gauge parameter dependence cancels prior to the expansion in m_D/T.

We do not list the renormalization counterterms necessary but mention that, as in the case of NNLO HTLpt QED [9], only systematic vacuum, mass, and coupling constant counterterms are necessary to renormalize the thermodynamic potential. The thermodynamic potentials listed below are gauge invariant. Full details of the Yang-Mills NNLO calculation will be presented elsewhere [13]; however, we note that the calculation is similar to the one presented in Ref. [9].

Leading order: The renormalized LO thermodynamic potential is [8]

$$\frac{\Omega_{\rm LO}}{\mathcal{F}_{\rm ideal}} = 1 - \frac{15}{2}\hat{m}_D^2 + 30\hat{m}_D^3 + \frac{45}{4}\left(\log\frac{\hat{\mu}}{2} - \frac{7}{2} + \gamma + \frac{\pi^2}{3}\right)\hat{m}_D^4, \tag{4}$$

where $\mathcal{F}_{\rm ideal} = -(N_c^2-1)\pi^2T^4/45$ is the free energy of an ideal gas of noninteracting gluons, γ is the Euler-Mascheroni constant, and we have introduced the dimensionless parameters $\hat{m}_D = m_D/(2\pi T)$ and $\hat{\mu} = \mu/(2\pi T)$.

Next-to-leading order: The renormalized NLO thermodynamic potential is [8]

$$\frac{\Omega_{\rm NLO}}{\mathcal{F}_{\rm ideal}} = 1 - 15\hat{m}_D^3 - \frac{45}{4}\left(\log\frac{\hat{\mu}}{2} - \frac{7}{2} + \gamma + \frac{\pi^2}{3}\right)\hat{m}_D^4 + \frac{N_c\alpha_s}{3\pi}\left[-\frac{15}{4} + 45\hat{m}_D - \frac{165}{4}\left(\log\frac{\hat{\mu}}{2} - \frac{36}{11}\log\hat{m}_D - 2.001\right)\hat{m}_D^2 + \frac{495}{2}\left(\log\frac{\hat{\mu}}{2} + \frac{5}{22} + \gamma\right)\hat{m}_D^3\right]. \quad (5)$$

Next-to-next-to-leading order: The renormalized NNLO thermodynamic potential is

$$\frac{\Omega_{\rm NNLO}}{\mathcal{F}_{\rm ideal}} = 1 - \frac{15}{4}\hat{m}_D^3 + \frac{N_c\alpha_s}{3\pi}\left[-\frac{15}{4} + \frac{45}{2}\hat{m}_D - \frac{135}{2}\hat{m}_D^2 - \frac{495}{4}\left(\log\frac{\hat{\mu}}{2} + \frac{5}{22} + \gamma\right)\hat{m}_D^3\right] + \left(\frac{N_c\alpha_s}{3\pi}\right)^2\left[\frac{45}{4}\frac{1}{\hat{m}_D} - \frac{165}{8}\left(\log\frac{\hat{\mu}}{2} - \frac{72}{11}\log\hat{m}_D - \frac{84}{55} - \frac{6}{11}\gamma - \frac{74}{11}\frac{\zeta'(-1)}{\zeta(-1)} + \frac{19}{11}\frac{\zeta'(-3)}{\zeta(-3)}\right) + \frac{1485}{4}\left(\log\frac{\hat{\mu}}{2} - \frac{79}{44} + \gamma + \log 2 - \frac{\pi^2}{11}\right)\hat{m}_D\right], \quad (6)$$

where ζ is the Riemann ζ function. Note that if the leading order Debye mass, $\hat{m}_{D,\rm LO}^2 = N_c\alpha_s/(3\pi)$, is used for the Debye mass in (6) we reproduce the known expansion of the Yang-Mills free energy up to order $\alpha_s^{5/2}$.

Mass prescription: The mass parameter m_D in HTLpt is, in principle, completely arbitrary. To complete a calculation, it is necessary to specify m_D as a function of α_s and T. Unfortunately, similar to NNLO HTLpt QED [9] the variational mass prescription gives a complex Debye mass. Here we equate the Debye mass used in HTLpt with the hard contribution to the Debye mass obtained using dimensional reduction [14], i.e., $m_D = m_E$ giving

$$\frac{\hat{m}_D^2}{\hat{m}_{D,\rm LO}^2} = 1 + \frac{N_c\alpha_s}{3\pi}\left(\frac{5}{4} + \frac{11}{2}\gamma + \frac{11}{2}\log\frac{\hat{\mu}}{2}\right). \quad (7)$$

Results.—In Figs. 1–3 we show the $N_c = 3$ pressure, entropy, and energy density scaled by their respective ideal gas limits as a function of T/T_c. The results at LO, NLO, and NNLO use Eq. (7) for the Debye mass in Eqs. (4)–(6), respectively. For the running coupling we used the three-loop running [15] and varied the renormalization scale by two around $\mu = 2\pi T$.

For the pressure, energy density, and entropy the convergence of the successive approximations to the Yang-Mills thermodynamic functions is improved over naive perturbation theory. For example, using the naive perturbative approach and comparing the full variation in both successive truncations and renormalization scale variation, one finds that at $T = 3T_c$ there is variation in the pressure of $0.69 \leq \mathcal{P}/\mathcal{P}_{\rm ideal} \leq 1.32$ [8], whereas using HTLpt there is only a variation of $0.74 \leq \mathcal{P}/\mathcal{P}_{\rm ideal} \leq 0.95$. Additionally, at NNLO we see that the $\mu = 2\pi T$ result for the pressure in Fig. 1 coincides with the lattice data down to $T \sim 3T_c$ and the energy density and entropy are compatible with lattice data down to $T \sim 2T_c$.

However, for all thermodynamic functions we find that the NNLO results represent a significant correction to the LO and NLO curves. This is unexpected since the LO and NLO bands overlap with one another at all temperatures shown. In addition, in NNLO HTLpt QED [9] such a large correction was not observed. For SU(3) Yang-Mills, in order for the LO, NLO, and NNLO bands to overlap one must go to temperatures $T \gtrsim 7T_c$. One may wonder if there

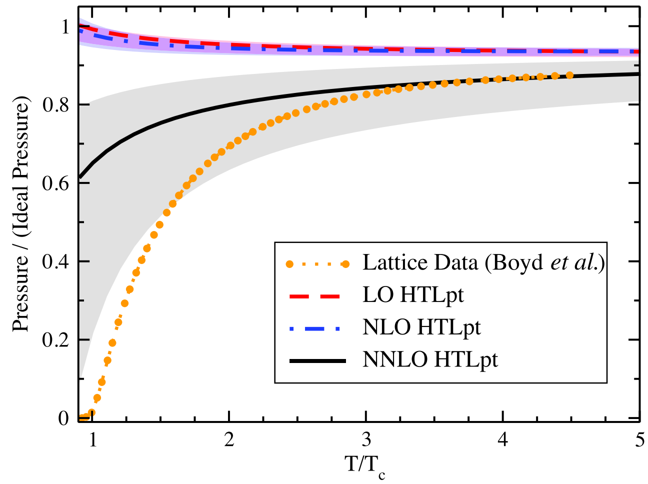

FIG. 1 (color online). Comparison of LO, NLO, and NNLO predictions for the scaled pressure with SU(3) pure-glue lattice data from Boyd *et al.* [4]. Shaded bands show the result of varying the renormalization scale μ by a factor of 2 around $\mu = 2\pi T$.

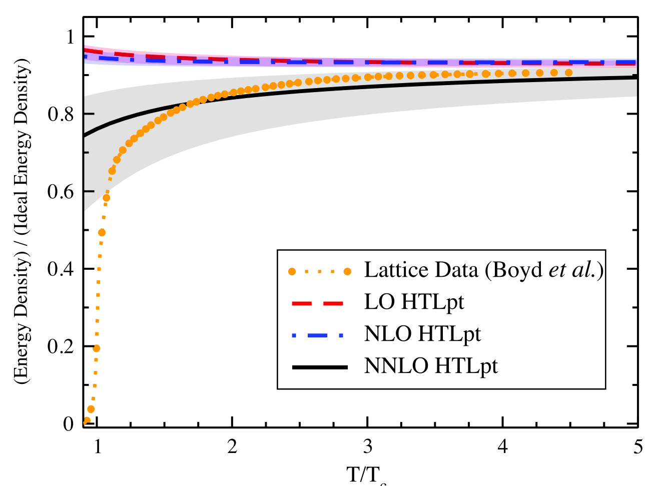

FIG. 2 (color online). Comparison of LO, NLO, and NNLO predictions for the scaled energy density with SU(3) pure-glue lattice data from Boyd *et al.* [4]. Shaded bands show the result of varying the renormalization scale μ by a factor of 2 around $\mu = 2\pi T$.

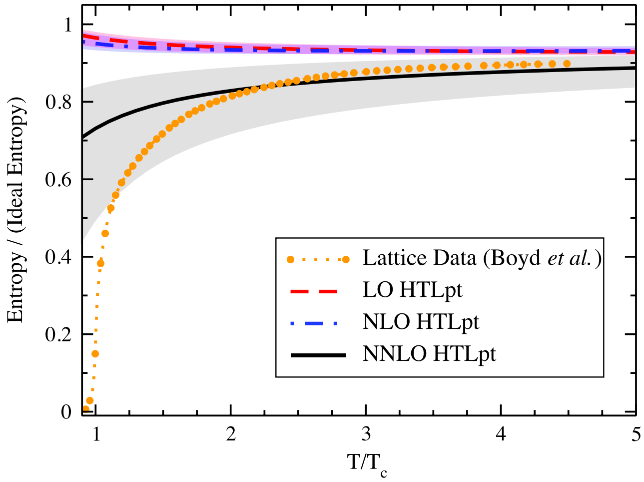

FIG. 3 (color online). Comparison of LO, NLO, and NNLO predictions for the scaled entropy with SU(3) pure-glue lattice data from Boyd *et al.* [4]. Shaded bands show the result of varying the renormalization scale μ by a factor of 2 around $\mu = 2\pi T$.

is an error in the NNLO thermodynamic potential; however, we are confident in this result because first, we reproduce the known perturbative expansion to order $\alpha_s^{5/2}$ in the weak-coupling limit using Eq. (6) and second, there were highly nontrivial cancellations of divergences using only systematically predicted counterterms during the renormalization procedure. We note that in scalar theories the corresponding reorganization has been pushed to N^{3}LO [11] where it has been shown that the N^{3}LO result is between the NLO and NNLO results, indicating an excellent pattern of convergence.

Conclusions and outlook.—In this Letter we have presented a new result for the NNLO thermodynamic functions for SU(N_c) Yang-Mills theory using the HTLpt reorganization. We compared our predictions with lattice data for $N_c = 3$ and found that HTLpt is consistent with available lattice data down to approximately $T \sim 3T_c$ in the case of the pressure and $T \sim 2T_c$ in the case of the energy density and entropy. These results are in line with expectations since below $T \sim 2$–$3T_c$ a simple "electric" quasiparticle approximation breaks down due to nonperturbative magnetic effects.

We found that at NNLO the variational solution for the Debye mass becomes complex and, as a result, we chose instead a NLO perturbative mass prescription. The complexity of the variational solution may be due to the truncation in m_D/T; however, checking this hypothesis will require future work. We also found that there was a large correction going from NLO to NNLO indicating that perhaps the result is not fully converged. Unfortunately, it is impossible to say how much a N^{3}LO calculation will affect things, so again future work is required.

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