Acquisition of Inertia by a Moving Crack

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We experimentally investigate the dynamics of ''simple'' tensile cracks. Within an effectively infinite medium, a crack's dynamics perfectly correspond to inertialess behavior predicted by linear elastic fracture mechanics. Once a crack interacts with waves that it generated at earlier times, this description breaks down. Cracks then acquire inertia and sluggishly accelerate. Crack inertia increases with crack speed v and diverges as v approaches its limiting value. We show that these dynamics are in excellent accord with an equation of motion derived in the limit of an infinite strip [M. Marder, Phys. Rev. Lett. **66**, 2484 (1991)].

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The motion of a crack is determined by balancing the energy that it dissipates with that which is externally supplied. In an elastic material, this energy is transported from a system's boundaries to the tip of a moving crack by means of the elastic fields that dynamically describe how stresses are distributed. In materials that are considered brittle, the transported energy is only dissipated within a very small region encompassing the crack's tip [1], called the process zone.

A crack will propagate when the elastic energy released by its extension G equals the energy dissipated per unit crack length Γ [2]. Once motion initiates, this energy balance is maintained dynamically. The classic framework that describes crack dynamics is called linear elastic fracture mechanics (LEFM). LEFM describes how linear elastic fields evolve in space and time to accommodate this balance.

The classic equation of motion for a crack (derived via LEFM) [1], however, is based on a number of key conditions. These include the assumption of an unbounded medium in which linear elasticity applies everywhere outside of the process zone. There is also an implicit assumption that *all* externally imposed forces can be mapped to tractions imposed on a crack's faces. When these conditions are met, the theory has been shown to beautifully describe both a crack's motion [3–5] and the form of the surrounding fields [6,7]. This description was shown to be valid [5] as long as a crack remains a single entity and does not undergo either microscopic (''microbranching'') [8] or macroscopic bifurcations. Under these conditions, cracks are predicted to have no inertia, where we define ''inertia'' as the (velocity dependent) coefficient coupling crack acceleration $\dot{v}$ to a crack's equation of motion.

In this Letter we describe how a crack behaves when some of these conditions are either not realized or break down. When allowed to interact with the (finite) boundaries of the medium, we will show that a crack acquires inertia. The inertia increases with crack velocity v until effectively becoming infinite as a crack's limiting velocity c_R is approached. We will also demonstrate that this behavior is in excellent quantitative agreement with an equation of motion derived in the framework of a crack propagating in an infinitely long strip [9].

Our experiments were performed using polyacrylamide gels which are transparent, brittle, incompressible elastomers. In these neo-Hookean materials the dynamics of rapid cracks are identical to those in more commonly used materials [10], with the advantage that they can be slowed by 3 orders of magnitude. As in [7,11], we use gels composed of 13.8% total monomer and 2.6% bis-acrylamide cross-linker concentrations. The shear ($\mu = 35.2 \pm 1.4$ kPa) and Young's ($E = 3\mu$) moduli of these gels yield respective shear, longitudinal and Rayleigh wave speeds of $c_S = 5.9 \pm 0.15$ m/s, $c_l = 11.8 \pm 0.3$ m/s and $c_R = 5.5 \pm 0.15$ m/s. Typical dimensions of the gels were ($X \times Y \times Z$) $125 \times 2b \times 0.2$ mm, where X, Y, and Z are, respectively, the propagation, loading, and thickness directions with b the system's half-width.

Experiments were performed as in [7,11] by imposing uniaxial (mode I) tensile loading via constant displacement in the vertical (Y) direction. Once a desired stress was reached, a guillotine was used to immediately initiate fracture at the sample's edge at $X = 0$, at a point $Y = b$, midway between the vertical boundaries. We varied the value of b to control the effective conditions of the system. To model an effectively infinite medium under constant tensile stress (''infinite medium'') we used $2b = 125$ mm and only considered crack dynamics for times $t < 2b/c_S$. At these times, shear waves, generated at fracture initiation and reflected back from the vertical boundaries, could not interact with the crack tip. An infinitely long strip (''infinite strip'') under conditions of constant displacement was modeled using $20 < 2b < 50$ mm, where only times *greater* than $2b/c_S$ and less than return times from the far lateral boundary were considered. Crack-tip locations were measured by a high speed camera set to $X \times Y$ resolutions of 1280×64 pixels (equivalent to 135.2×6.8 mm) for experiments in infinite media, and 1280×210 pixels (57.5×19 mm) for experiments in infinite strips. In each type of experiment, successive photographs were taken at 7200/2400 frames/s with exposure times of 4.4/3 μs. Multiple exposures were utilized, when needed. In all of the experiments performed, the microbranching instability was suppressed (as in [11]) by setting the gel thickness to 160–220 μm. This enabled single-crack states to propagate over a wide velocity range, $0.20c_R < v < 0.95c_R$. $v > 0.9c_S$ were not considered, as they correspond to oscillating cracks [11].

We now consider a crack in an infinite medium under a constant tensile stress σ_∞ imposed at the system's vertical boundaries. As the crack propagates, it generates a moving elastic field that radiates away from the crack's tip. As long as these waves are not reflected back to the crack tip, this configuration corresponds [8] to a crack propagating within an effectively infinite medium that is loaded by a constant stress along its faces. The corresponding energy release rate G is given by [1]:

$$G(l, v) = \frac{1-\nu^2}{E} K_I^2(l, v) A_I(v), \qquad (1)$$

where ν and E are the material's Poisson ratio and Young's modulus, and $A_I(v)$ is a universal function of the crack's instantaneous speed. The stress intensity factor $K_I(l, v) = K_s(\sigma_\infty, l)k(v)$ is the coefficient of the $1/\sqrt{r}$ stress singularity at the crack's tip. The function $K_s(\sigma_\infty, l)$ incorporates the loading conditions and instantaneous crack length l, whereas $k(v)$ is another universal function predicted by LEFM [1]. For "infinite medium" conditions [12], $K_s(\sigma_\infty, l) = \sqrt{\frac{8}{\pi}}\sigma_\infty\sqrt{l}$, and LEFM predicts that

$$G(l, v) \approx \frac{1-\nu^2}{E}\frac{8l}{\pi}\sigma_\infty^2(1 - v/c_R). \qquad (2)$$

Once $\Gamma(v)$ is known, Eq. (2) yields the equation of motion for a single-crack through energy balance $G(l, v) = \Gamma(v)$. In an infinite system under constant stress loading conditions, energy balance requires $G(l, v)$ to remain finite as $l \to \infty$. Equation (2) tells us that this balance is dynamically maintained by the $1 - v/c_R$ factor. Hence v must always accelerate monotonically to its limiting velocity c_R as a crack lengthens without limit.

We test the equation of motion for these conditions in a number of experiments in which σ_∞ was varied. $\Gamma(v)$ for this material was independently determined [7] by measuring the (LEFM-predicted) parabolic curvature of the crack tip as a function of v at $\sim$1 mm scales [13]. The results of these experiments, presented in Fig. 1, reveal perfect correspondence with LEFM predictions over an unprecedented range of v. Another stringent test of LEFM predictions is provided in the inset of Fig. 1, where all of these measurements are shown to collapse onto the predicted universal function $k(v)$.

The agreement with the Eq. (1) breaks down once the experimental conditions modeling an infinite system are invalidated. As we show in Fig. 2, this occurs when the waves generated by the crack tip upon initiation interact with the crack tip after reflection from the sample's vertical boundaries. The sharp break with the predicted equation of motion typically occurs at $t \sim 2b/c_S$. Beyond this point, a crack increasingly interacts with its own prior "history," as the reflected elastic waves increasingly affect the energy flux into the crack tip. When this happens, the LEFM predictions encapsulated in Eq. (2) certainly fail. It is unclear whether these complex spatially and temporally varying loading conditions can even be incorporated into the framework leading to Eq. (1).

Once t significantly exceeds $2b/c_S$, v approaches steady-state values that can be significantly lower than c_R, as Fig. 3(a) demonstrates. How can a steady-state value of $v < c_R$ be understood? For $l \gg b$ we approach the limit of an infinite strip loaded with a finite amount of elastic

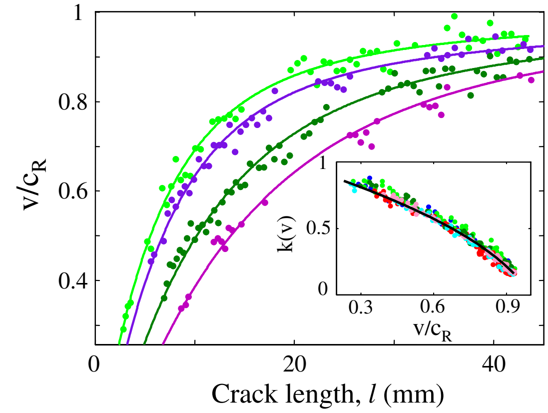

FIG. 1 (color online). Measured v as a function of the crack length l (circles) compared to LEFM predictions (solid lines) for values (top to bottom) of $\sigma_\infty = 15.7$ kPa, 14.9 kPa, 12.4 kPa, and 10.4 kPa. All measurements are for single cracks propagating in "infinite medium" conditions ($2b = 125$ mm). (inset) Comparison of measured $k(v)$ [24] (circles) to the LEFM-predicted universal function: $k(v) \approx (1 - v/c_R)/\sqrt{1 - v/c_l}$ (solid line).

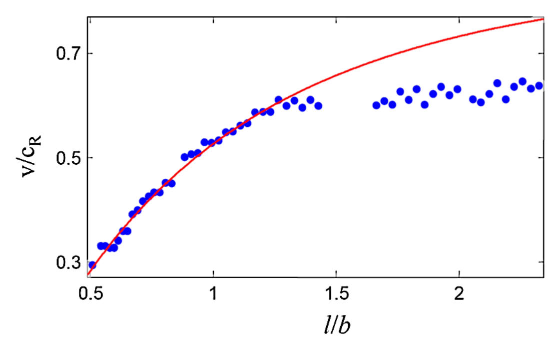

FIG. 2 (color online). Comparison of the measured crack velocity in a sample of dimensions $120 \times 30 \times 0.2$ mm (circles) to the LEFM approximation for an infinite medium (solid line). The sharp divergence occurs at $t \sim 2b/c_s$, when the returned waves from the vertical boundaries interact with the crack tip. Here, $\sigma_\infty = 12.2$ kPa.

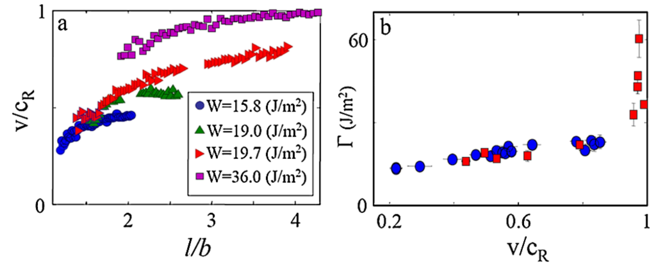

FIG. 3 (color online). (a) Typical crack velocity measurements in a strip geometry $120 \times 30 \times 0.2$ mm. Note that each experiment approaches a different final velocity $v_{\rm final}$ as the energy density W stored in the strip is changed (legend). (b) Fracture energy $\Gamma(v)$, calculated via $\Gamma(v_{\rm final}) = W$ (squares) compared to values of $\Gamma(v)$ measured by means of crack-tip curvature [7] (circles). Careful analysis of crack dynamics in a strip will show (Fig. 4) that the large increase in Γ is spurious.

energy per unit area W. In this limit, the system is translationally invariant for a crack moving at a constant v, as the energy W released at $l \to \infty$ is precisely compensated by the amount dissipated, $\Gamma(v)$. Hence, in this limit a crack should reach a steady-state velocity determined by the energy balance condition: $W = \Gamma(v)$. This argument is independent of any of the details of the system, on the condition that v reaches a steady state. This configuration has been successfully used to measure $\Gamma(v)$ in a number of different materials [14–16].

In Fig. 3(b) we compare the values of $\Gamma(v)$ measured via crack-tip curvature of dynamic cracks [7] with those obtained in the strip geometry for cracks propagating at approximate steady-state velocities. W, in each strip experiment, was measured by integrating the stress-strain curve prior to crack initiation. For $0.20 < v < 0.85 c_R$ we see that the two measurements are in very good quantitative agreement. Figure 3(b), however, shows that, for $v \to c_R$, the apparent values of $\Gamma(v)$ obtained in the strip measurements appear to diverge.

This divergence of $\Gamma(v)$ is not real. The key to understanding it lies in the energy balance condition for a strip, $W = \Gamma(v)$. Whereas $\Gamma(v)$ is a characteristic function of a given material, the energy density W is wholly governed by the value of σ_∞ applied to the strip. It is, thereby, entirely possible to "overstretch" the strip so that $W \gg \Gamma(v)$ for *any* crack velocity. Why do we empirically observe nearly steady-state motion under *extremely* overstretched conditions? Under these conditions, we find that seemingly negligible crack acceleration leads to anomalously high apparent values of Γ.

In an *infinite* medium, when extreme stresses are imposed, Eq. (1) compensates for any "excess" elastic energy by increasing the amount of kinetic energy (i.e., motion) within the medium. This is the physical mechanism that provides a limiting crack velocity. When an unlimited amount of energy is available in an infinite medium, LEFM converts *all* of the imposed elastic energy to motion, while still retaining energy balance. Is there an equivalent mechanism when conditions of an infinite strip are realized?

Marder [9] performed a perturbative derivation of an equation of motion for a crack in an infinite strip, where the dimensionless acceleration $b\dot{v}/c_l^2$ was assumed to be small. Under these conditions:

$$G = W\left(1 - \frac{b\dot{v}}{c_l^2} f(v)\right) \approx W\left(1 - \frac{b\dot{v}}{c_l^2} \frac{1}{(1 - (\frac{v}{c_R})^2)^2}\right). \quad (3)$$

In sharp contrast to the assumptions leading to Eq. (1), this derivation assumes that the crack is *always* in approximate equilibrium with its vertical boundaries. When $\Gamma(v) = W$, the system can attain any compatible steady-state velocity v. Once *overstressed* [i.e., $W > \Gamma(v)$ for all v] a crack, as in Eq. (1), will accelerate to the same limiting velocity c_R. Here, however, the similarities between Eq. (1) and (3) end. Whereas a crack in an infinite medium has *no* inertia (i.e., no $\dot{v}$ dependence), Eq. (3) predicts precisely the opposite effect. A crack's interaction with its "past," via the elastic waves continuously reflected by the vertical boundaries, imparts it with a v-dependent inertia that, in fact, *diverges* as $v \to c_R$.

In Fig. 4(a) we perform a quantitative test of the velocity dependent function $f(v)$ in Eq. (3). The predicted value of $f(v)$ (given in [17]) was compared to the derived value, $f(v) \equiv [c_l^2/(b\dot{v})](1 - \Gamma(v)/W)$, for 8 different experiments over a wide range of W ($27.8 - 55.5$ J/m^2) and b (10–25 mm). The agreement with theory is very good and further improves when $(b\dot{v})/c_l^2 < 0.01$, in accordance with the perturbative expansion. Surprisingly, the measured values of $f(v)$ agree well with predictions for $(1 - \Gamma/W) \sim O(1)$, a range well beyond the expected region of validity of Eq. (3) [9].

In Fig. 4(b) we perform a direct comparison between the predicted [using Eq. (3)] and measured values of $\Gamma(v)$. The excellent agreement for all values of v indicates both the

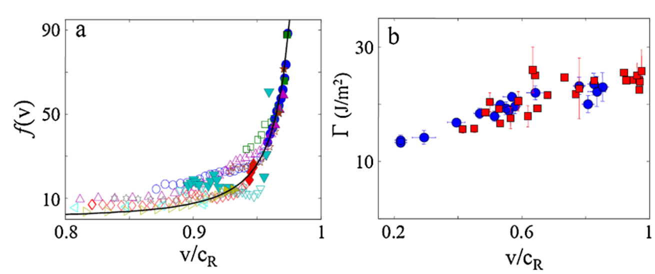

FIG. 4 (color online). (a) Comparison of $f(v)$ in Eq. (3) as calculated from [17] (solid line) with the values derived using $f(v) = c_l^2/(b\dot{v})(1 - \Gamma(v)/W)$ (symbols). Filled symbols indicate measurements where the dimensionless acceleration, $(b\dot{v})/c_l^2 < 0.01$. (b) Measured fracture energy $\Gamma(v)$ using Eq. (3) (squares) compared to $\Gamma(v)$ as measured via crack-tip curvature [7] (circles). The large corrections to G due to the dimensionless acceleration in Eq. (3) at high velocities indicate the effect of large crack inertia.

validity of Eq. (3) together with the important role that crack inertia can play at high crack velocities. We note that crack dynamics in samples which are far from a formal strip configuration (e.g., Fig. 2), are still well described by Eq. (3) for $t > 2b/c_S$.

In summary, we have verified, for an unprecedented range of crack velocities, that the classic equation of motion of a single crack predicted by LEFM works extremely well, as long as the loading conditions for which it was developed are valid. One of the more surprising features of this theory is that of an inertialess crack. We have verified that crack dynamics are indeed inertialess, until the point where a crack is able to interact with its prior history. At this stage, crack dynamics sharply change. We then showed that, in general, when a crack is able to *continually* interact with its past, crack inertia has a dominant effect on dynamics. In this case, we demonstrated that theoretical predictions [9,17] for crack dynamics in an infinite strip are in excellent agreement with measurements, even beyond their formal region of validity. Here, crack inertia was seen to diverge as $v \to c_R$. This divergence is akin to the mass divergence of particles in special relativity as they approach the speed of light.

Matching steady-state crack velocities to the energy stored per unit length in a strip, is a common method to measure the fracture energy [14–16]. When used, there is always a tacit assumption that the energy balance inherent in $\Gamma(v) = W$ is valid. This is often true, as in the case where the microbranches instability causes $\Gamma(v)$ to be a strongly increasing function of v [14]. On the other hand, one must exercise care in applying this technique, as in an overstressed system at high velocities, crack inertia diverges and vanishingly small accelerations can still be quite far from "steady-state" propagation.

We have shown that there are two distinct classes of propagation dynamics for simple (nonbifurcating) cracks, even when the simplest material constitutive law (linear elasticity) is assumed. These describe crack dynamics driven via elastic fields that couple remote loading to a crack's tip. Are there other dynamic solutions? Apparently yes. An entirely new class of supersonic solutions in tensile fracture are predicted once the elastic energy density is so large in the vicinity of a crack's tip, that energy does not need to be transported from remote locations to satisfy energy balance [18–20].

Is the emergence of inertia a general consequence of a crack's interaction with its past? This is an interesting and very much open question. Here we looked at an extreme case, where these interactions are continuously taking place. A crack can also undergo transient self-interactions when a crack front impinges on a localized material inhomogeneity. Such "inertial" memory effects were indeed observed [21] as "ring-downs" of the perturbed crack fronts that continue long after the inhomogeneity was left behind. Recent theoretical work [22] further suggests that such inertial effects could also drive oscillatory instabilities of cracks at high velocities [11,23].

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