Zero Lag Synchronization of Chaotic Systems with Time Delayed Couplings

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Zero-lag synchronization (ZLS) between chaotic units, which do not have self-feedback or a relay unit connecting them, is experimentally demonstrated for two mutually coupled chaotic semiconductor lasers. The mechanism is based on two mutual coupling delay times with certain allowed integer ratios, whereas for a single mutual delay time ZLS cannot be achieved. This mechanism is also found numerically for mutually coupled chaotic maps where its stability is analyzed using the Schur-Cohn theorem for the roots of polynomials. The symmetry of the polynomials allows only specific integer ratios for ZLS. In addition, we present a general argument for ZLS when several mutual coupling delay times are present.

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The synchronization of chaotic systems is a fundamental phenomenon in nonlinear dynamics [1]. A set of nonlinear units, mutually coupled by a function of their internal variables, can synchronize to a common chaotic trajectory [2–5]. Chaos synchronization has interesting applications for secure communications since secret messages can be transmitted by the chaotic signal [6–9]. The receiving unit can decode it, but it is difficult, if not impossible, to extract the message from the irregular signal for an illegal listener. This phenomenon has been realized with electronic circuits and chaotic lasers [4–6,8,10–12]. Typically, the transmitted signal between two units that establishes synchronization has a time delay [13–17]. For chaotic semiconductor lasers coupled over a distance greater than about 1 m, the delay time is larger than most of the internal time scales of the laser. The synchronization can have various properties, the chaotic units can synchronize without time shift [zero-lag synchronization (ZLS)] [18–22], the receiving unit can predict the trajectory of the sending unit (anticipated synchronization) [23], or the trajectories can be related by a nonlinear function (generalized synchronization). In particular, isochronal synchronization, or ZLS, which has been discussed in both a communications context and with regards to neural networks, is a counterintuitive phenomenon. Although the transmitted signal is received with an arbitrary long delay time, the chaotic units synchronize without time delay. Thus two neurons can synchronize their irregular spiking events without time shift although their action potentials are transmitted with a considerable time delay [24,25].

It is not obvious how to achieve ZLS. Recently it has been shown that units which are chaotic when the couplings are removed cannot be synchronized when the coupling delay is much larger than the internal time scales including the Lyapunov time [26]. Hence, the chaotic trajectories of the network have to be generated by the mutual interactions to achieve ZLS. But even in the latter case it has been reported that two chaotic lasers cannot be synchronized with only one mutual coupling isochronally [18,27]. Instead, two lasers coupled by one mutual coupling synchronize with a time lag, so one of the lasers is lagging behind the other, as shown in [27]. In order to achieve ZLS, each of the lasers also needs a self-feedback or the coupling between them has to be via a relay unit or a third driving unit. In addition, when adding self-feedback, either the self-feedback delay time has to be carefully adjusted to equal the coupling delay time or multiple self-feedback delay times have to be used [28]. In this Letter we show that it is possible to achieve ZLS of two chaotic semiconductor lasers without a relay unit or self-feedbacks. Our experiments demonstrate that ZLS is achieved when multiple coupling delay times with well-defined ratios are used. The specific values of the ratio can be derived from the theory of the roots of polynomials which gives analytical conditions for ZLS for coupled Bernoulli maps. When the delay ratios are not equal to these special values the synchronization is significantly reduced and even eliminated as measured by the cross correlation of the temporally unshifted intensity traces of the two lasers.

The experimental setup is shown in Fig. 1. Two similar Fabry-Perot semiconductor diode lasers emitting in the visible range near 656 nm wavelength are coupled bidirec-

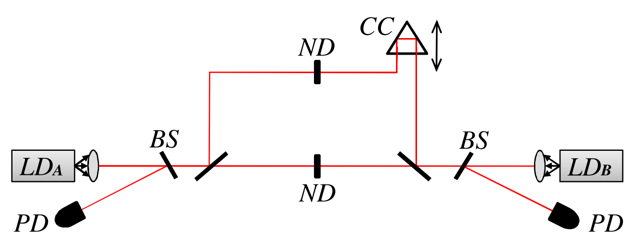

FIG. 1 (color online). Experimental setup: two semiconductor lasers coupled by two bidirectional couplings with different optical path lengths. LD, laser diode; PD, photodiode; BS, beam splitter; ND, neutral density filter; CC, corner cube.

tionally by two optical paths with different time delays—one constant at a time delay of $\tau_1 = 14.12$ ns and one made variable by using a corner-cube mirror on a computer controlled precision translation stage. When subjected to mutual feedback, the lasers display chaotic behavior, consisting of very short and random spiking of the laser intensity. The two lasers are temperature tuned to operate at nearly identical wavelengths while their injection current to threshold current ratios are maintained nearly equal at $p = I/I_{th} = 1.2$. The coupling strength is controlled by two neutral density (ND) filters, and a beam splitter near each laser couples a small portion of the beam into fast photodetectors, sampled by a digital oscilloscope with 8 bit resolution and a maximal sampling rate of 40 GHz. We found it convenient to perform the experiments in a synchronization regime where total laser intensity breakdowns take place, commonly referred to as the low frequency fluctuation (LFF) regime [29]. In this regime, intensity breakdowns of both lasers are observed, and the system temporarily desynchronizes as shown in Fig. 2, similar to what has been reported in [30]. In order to avoid these irregularities, we divide the chaotic waveform into segments of 10 ns length, calculate the correlation coefficient for each segment using the cross correlation function, and calculate the average correlation from half of those segments having the highest correlation [21].

The Lang-Kobayashi (LK) equations are known to be a good model of coupled semiconductor lasers [31]. Hence we have investigated the scenario of Fig. 1 by simulating the LK model with the parameters of Ref. [21]. In correspondence to our experiments the intensity in the simulations was averaged over 1 ns and the cross correlation was calculated in windows of 10 ns for 1 μs. Then, as in the experiment, the highest 50% of the correlation values were used in calculating the average correlation so as to avoid the severe correlations drops in the LFFs. In Fig. 3 we compare the measured and calculated correlation coefficient of the two lasers for a fixed value of $\tau_1 = 10$ ns and varying values of τ_2. Both delay loops have the same coupling strength $\sigma_1 = \sigma_2 = 50\ \mathrm{ns}^{-1}$. The agreement between simulation and experimental results is reasonable. Representative data, showing 10 ns long chaotic trajectories for both lasers at different τ_2/τ_1 ratios, are presented in Fig. 4. The figure clearly demonstrates ZLS for the two chaotic trajectories at $\tau_2/\tau_1 = 2$ [4(a)] and the lack of ZLS at a ratio of $\tau_2/\tau_1 = \frac{5}{3}$ [4(d)]. Our experimental and numerical data of Fig. 3 show ZLS for $\tau_2 = 2\tau_1$. For $\tau_2 = \frac{3}{2}\tau_1$ we still get a significant ZLS correlation value of 0.89 in the simulation and 0.77 in the experiment. For ratios of $\frac{5}{4}, \frac{4}{3}, \frac{5}{2}$ we observe correlation values $C > 0.5$ while for $1, \frac{5}{3}$, 3 the correlation values drop to $C < 0.4$. Based on these results we claim that ZLS is not possible if $\tau_2/\tau_1 = p/q$ is a ratio consisting of relatively prime numbers p, q being odd. We provide qualitative and even quantitative arguments to sustain our conjecture.

As we already noted, two mutually coupled chaotic lasers do not synchronize isochronally if coupled by a single time delay τ_1. At least two coupling delay times are needed to achieve ZLS. A simple qualitative argument gives the necessary conditions for the coupling delay ratios. We assume that in order to synchronize with another unit at distance τ, a chaotic unit A_t needs information not only on the time-delayed signal of $B_{t-\tau_i}$ but also on its own time-delayed state $A_{t-\tau_i}$. If we assume that B acts to some extent like a mirror, the information transmitted by A returns to itself after time 2τ. Thus, if we add an additional signal with delay 2τ both units receive the same information after time 2τ, namely (with $\tau_i = 2\tau$) the signals $A_{t-2\tau}$ and $B_{t-2\tau}$ and they may be able to synchronize. Although the lasers are not perfect mirrors and the signal is slightly changed in the interchange between the units, we claim that the changes are small enough to allow sustained synchronization. Note that a small change is even desired; otherwise the trajectory would not be chaotic but periodic. But of course the more often the signals are exchanged, the larger are the changes imprinted on the returning signal.

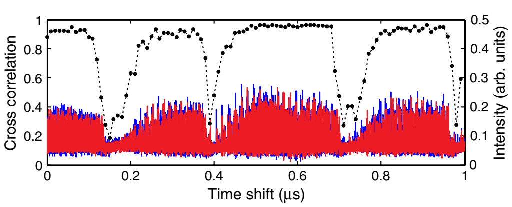

FIG. 2 (color online). Recording of the time dependent intensities of two zero-lag synchronized lasers (solid lines) and the calculated cross correlation of the two lasers (black circles). Notice the desynchronization periods associated with power dropouts (LFFs).

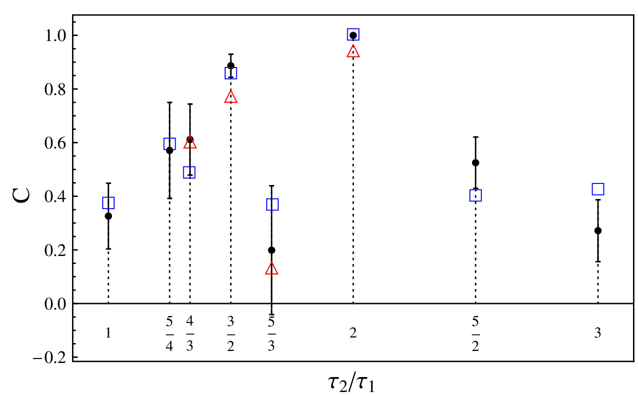

FIG. 3 (color online). Cross correlation coefficient for two chaotic units which are mutually coupled via two time-delayed couplings with τ_1 and τ_2. Dots: Lang-Kobayashi simulation with $\tau_1 = 10$ ns and τ_2 is varied, $\sigma = 50\ \mathrm{ns}^{-1}$. Triangles: Experimental data with same parameters as simulation. Squares: Data from simulation with Bernoulli units with $a = 0.45$, $b = c = 0.3$; see Eq. (2). Error bars mark the standard deviation from several simulations of the Lang-Kobayashi equations.

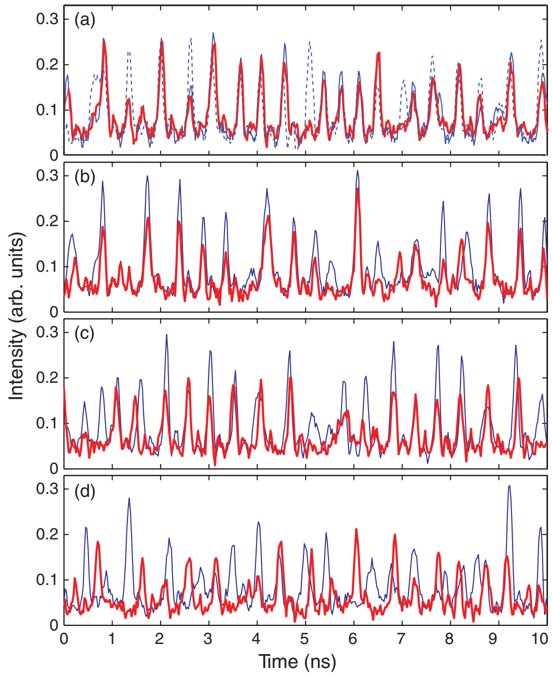

FIG. 4 (color online). A 10 ns recording of the time dependent intensity of two semiconductor lasers (thin blue and thick red). (a) $\tau_2/\tau_1 = 2$, (b) $\tau_2/\tau_1 = \frac{3}{2}$, (c) $\tau_2/\tau_1 = \frac{4}{3}$, (d) $\tau_2/\tau_1 = \frac{5}{3}$. To demonstrate the slow decorrelation of the intensities with increasing time we show in panel (a) (dashed line) the intensity of the laser after a delay of $4\tau_1$. Corresponding correlation values can be seen in Fig. 3.

This is the reason why for large ratios like $\tau_2 = 20\tau_1$ ZLS is not obtainable. The same argument gives conditions for which ZLS with more than two delay times is possible (see below) and it also explains ZLS for configurations with self-feedback or with a relay unit. Following these arguments, two units without self-feedback attain ZLS when coupled by three mutual couplings as well, if, e.g., $\tau_1 + \tau_2 = \tau_3$. Indeed our simulations of a Lang-Kobayashi system with three time delays, e.g., $\tau_1 = 30$ ns, $\tau_2 = 50$ ns, $\tau_3 = 80$ ns yield correlation values close to 1. This is also consistent with the results of [28], where we found ZLS for multiple self-feedback times τ_{di} and multiple coupling times τ_{cj} that obeyed $\sum_{i=1}^{N_s} n_i\tau_{di} + \sum_{j=1}^{N_m} m_j\tau_{cj}$ with restricted values of m_j and n_i. Our qualitative argument indicates that ZLS with several mutual coupling delays is a general phenomenon. Hence we present an analytically solvable model for which the conditions for ZLS are derived from the symmetry of polynomials. We consider two iterated Bernoulli maps $f(x) = (\alpha x)\,\mathrm{mod}\,1$, which are coupled by two time-delayed terms analog to the setup of the semiconductor lasers,

$$x_t = (1-\epsilon)f(x_{t-1}) + \epsilon\sigma f(y_{t-\tau_1}) + \epsilon(1-\sigma)f(y_{t-\tau_2}),$$
$$y_t = (1-\epsilon)f(y_{t-1}) + \epsilon\sigma f(x_{t-\tau_1}) + \epsilon(1-\sigma)f(x_{t-\tau_2}).$$

Agreement between the results of this model and the results of a system with coupled chaotic semiconductor lasers can be seen in Fig. 3. Obviously, in this model the synchronized trajectory $x_t = y_t$ is a solution of the iteration $x_t = (1-\epsilon)f(x_{t-1}) + \epsilon\sigma f(x_{t-\tau_1}) + \epsilon(1-\sigma)f(x_{t-\tau_2})$, which is chaotic for $\alpha > 1$. Note that the two trajectories x_t and y_t are completely synchronized without any time shift, although the signal is transmitted with two time delays τ_1 and τ_2.

The stability of this solution is checked by adding a small perturbation $x_t = y_t + \delta_t$. In the limit of infinitely small δ_t one can ignore the jump generated by the sawtooth shape of $f(x)$, and the perturbations obey the linear equation

$$\delta_t = (1-\epsilon)\alpha\delta_{t-1} - \epsilon\sigma\alpha\delta_{t-\tau_1} - \epsilon(1-\sigma)\alpha\delta_{t-\tau_2}. \quad (1)$$

The ansatz $\delta_t = z^t$ gives the polynomial equation (for $\tau_2 > \tau_1$)

$$z^{\tau_2} - az^{\tau_2-1} - bz^{\Delta} - c = 0, \quad (2)$$

with the coefficients $a = (1-\epsilon)\alpha$, $b = -\epsilon\sigma\alpha$, $c = -\epsilon(1-\sigma)\alpha$, and the time difference $\Delta = \tau_2 - \tau_1$. Note that to ensure a chaotic trajectory of the system, $a + b + c \geq 1$. Whereas the parameters in (1) are chosen so that $a + b + c = \alpha \geq 1$, for the general parameters in (2) the condition $a + b + c \geq 1$ is kept in simulations for Fig. 3. ZLS is stable when all roots of the polynomial (1) lie inside the unit circle of the complex plane. This problem is well studied in control theory [32], and the Schur-Cohn theorem [33] allows us to calculate the region of parameters (a, b, c) where ZLS is stable. Since we want to compare the results of the Bernoulli system with chaotic lasers, we consider the case where an isolated unit is not chaotic, $a = (1-\epsilon)\alpha < 1$. Hence, chaos is generated by the mutual couplings. In fact, for $a = 0$ we immediately obtain the following general results ($\tau_2 \geq \tau_1$). Only the ratio $\tau_2/\tau_1 = p/q$ determines ZLS, where p and q are relatively prime integers. Consequently, ZLS is stable even in the limit of infinitely long delay times. Our main result is, if p and q are odd integers, including a single delay $\tau_1 = \tau_2$, ZLS is excluded for any parameters b and c for which the system is chaotic. To see this, note that the transformation $w = \pm z^n$ for any positive integer n defines a new polynomial. When all roots of the z polynomial lie inside the unit circle, then all roots of the w polynomial lie inside the unit circle as well. Correspondingly, the same result can be obtained for $a \neq 0$ and $\tau_i \to \infty$ with τ_1/τ_2 being constant. When p and q are not both odd, the region of coefficients b and c for which ZLS is stable can be calculated analytically using the Schur-Cohn theorem [33]. Figure 5 shows the results for $(p, q) = (2, 1), (3, 2), (4, 3)$, and $(5, 2)$. Only the regions for which the system is chaotic are shown ($-b - c > 1$). Obviously, the parameter region for ZLS is largest for $\tau_2 = 2\tau_1$, and it shrinks with increasing values of p and q. In fact, when $\Delta = \tau_2 - \tau_1$ is finite we can prove that this parameter region decreases to zero when τ_2 and τ_1 diverge

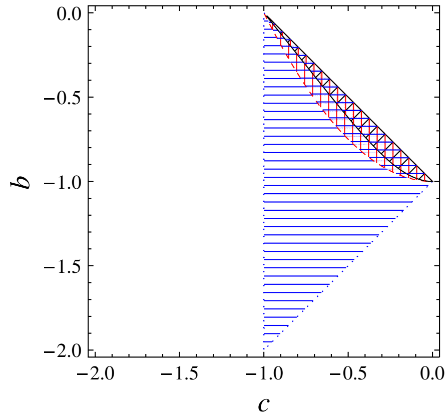

FIG. 5 (color online). ZLS for two units coupled by two time-delayed couplings (τ_1, τ_2). Only the chaotic region is shown. Region with horizontal lines, $\tau_2/\tau_1 = 2$; vertical lines, $\tau_2/\tau_1 = 3/2$; diagonal lines, $\tau_2/\tau_1 = 5/2$.

to infinity. In agreement with our analytic results, ZLS in chaotic lasers are most easily found for $\tau_2 = 2\tau_1$ and less easily for other ratios as the coupling parameters have to be chosen more carefully. In addition, the values of the cross correlation obtained numerically for the Bernoulli system are similar to the ones from chaotic semiconductor lasers, as shown in Fig. 3.

For the Bernoulli maps the local Lyapunov exponents are constant. Therefore the stability of the synchronization manifold is given by the roots of the polynomial, Eq. (2). For other maps, the coefficients of the linearized equations depend on time. We have checked for two logistic maps whether the predictions derived from Eq. (2) hold for time dependent coefficients as well. When τ_2/τ_1 is a ratio of two relatively prime odd integers, chaotic ZLS should not be possible. In fact, e.g., for $\tau_2/\tau_1 = \frac{5}{3}$ we find ZLS, but the system generated a periodic trajectory. Chaos is ruled out by symmetry, and periodic orbits are stabilized, similar to chaos control [14,34].

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