Three-Loop Static Potential

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(Received 25 November 2009; published 15 March 2010)

We compute the three-loop corrections to the potential of two heavy quarks. In particular, we consider in this Letter the purely gluonic contribution which provides, in combination with our previous fermion corrections, the complete answer at three loops.

DOI: 10.1103/PhysRevLett.104.112002 PACS numbers: 12.38.Bx

The potential between two heavy quarks constitutes a fundamental quantity in quantum chromodynamics (QCD). It enters in a variety of physical processes like the threshold production of top quark pairs and the description of charm and bottom quark bound states. Furthermore, it is crucial for the understanding of fundamental quantities of QCD, such as confinement. (See Ref. [1] for a recent review.)

The idea to describe a bound state of heavy colored objects in analogy to the well-established hydrogen atom goes back to the middle of the 1970s [2]. Shortly afterwards, about 30 years ago, one-loop radiative corrections were evaluated in the works [3,4]. It took almost 20 years until the next order became available [5–7], which, at that time, was a heroic enterprise. The two-loop corrections turned out to be numerically quite important, which triggered several investigations to go beyond. At the end of 2009, the fermionic corrections to the three-loop static potential were completed [8–10]. In this Letter we report on the pure gluonic part, which completes the three-loop corrections to the static potential.

We present our results for the static potential in momentum space where it takes the form

$$V(|\vec{q}|) = -\frac{4\pi C_F \alpha_s(|\vec{q}|)}{\vec{q}^2}\left[1 + \frac{\alpha_s(|\vec{q}|)}{4\pi} a_1 + \left(\frac{\alpha_s(|\vec{q}|)}{4\pi}\right)^2 a_2 + \left(\frac{\alpha_s(|\vec{q}|)}{4\pi}\right)^3 \left(a_3 + 8\pi^2 C_A^3 \ln\frac{\mu^2}{\vec{q}^2}\right) + \cdots\right]. \quad (1)$$

Here, $C_A = N_c$ and $C_F = (N_c^2 - 1)/(2N_c)$ are the eigenvalues of the quadratic Casimir operators of the adjoint and fundamental representations of the $SU(N_c)$ color gauge group, respectively, and α_s denotes the strong coupling in the $\overline{\rm MS}$ scheme. The one- and two-loop coefficients a_1 [3,4] and a_2 [5–7,11] are given in Eq. (4) of Ref. [8], where the higher order terms in ϵ, necessary for the three-loop calculation, are also presented. In Eq. (1) we identify the renormalization scale μ^2 and the momentum transfer $\vec{q}^2$. The complete dependence on μ can easily be restored with the help of Eq. (2) of Ref. [8].

A new feature of the three-loop corrections to $V(|\vec{q}|)$ is the appearance of infrared divergences [12], which is represented by the $\ln(\mu^2/\vec{q}^2)$ term in Eq. (1). It has been evaluated for the first time in Refs. [13,14] (see also Ref. [15]); in Eq. (1) we adopt the $\overline{\rm MS}$ scheme which has been used in Ref. [14]. Let us mention that the infrared divergence cancels in physical quantities after including the contribution where so-called ultrasoft gluons interact with the heavy-quark antiquark bound state. An explicit result can be found, e.g., in Ref. [14], where the cancellation has been demonstrated in order to arrive at the measurable energy levels of the heavy-quark system. We note in passing that higher order logarithmic contributions to the infrared behavior of the static potential have been computed in Refs. [16,17].

Before presenting our results for a_3, let us provide some technical details. We generate the four-point quark antiquark amplitudes with the help of QGRAF [18]. Some sample diagrams up to three-loop order are shown in Fig. 1. In the next step they are processed further with Q2E and EXP [19,20], where a mapping to the diagrams of Fig. 2 is achieved. The mapping to two-point functions is possible since the only dimensionful quantity in our problem is given by the momentum transfer between the quark and the antiquark. Although there is only one mass scale in our problem, technical complications arise from the simultaneous presence of static lines (zigzag lines) and relativistic propagators (solid lines), which significantly increases the complexity of the reduction to master integrals. For this task we employ the program package FIRE [21] in order to

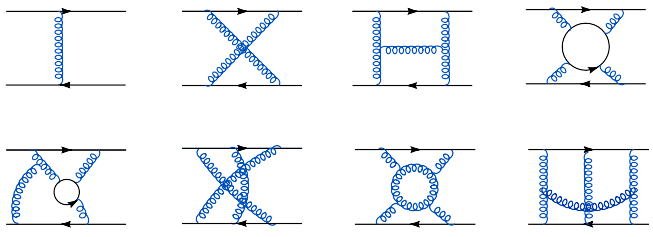

FIG. 1 (color online). Sample diagrams contributing to the static potential at tree-level, one-, two-, and three-loop order. Solid lines and curly lines represent quarks and gluons, respectively. In the case of closed loops the quarks are massless; the external quarks are heavy and treated in the static limit.

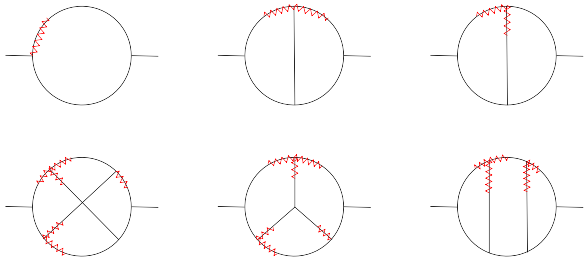

FIG. 2 (color online). One-, two-, and three-loop diagrams. The solid line stands for massless relativistic propagators and the zigzag line represents static propagators.

achieve a reduction to about 100 basic integrals, so-called master integrals. The latter have to be evaluated in an expansion in ϵ, which we achieve with the help of the Mellin-Barnes method (see, e.g., Refs. [22–25]). We managed to compute all the necessary coefficients of the ϵ expansion of the master integrals analytically with the exception of terms of order ϵ^1 of the three diagrams shown in Fig. 3. Results for the master integrals as well as more details on their evaluation will be published elsewhere. As a crucial tool providing very important numerical cross-checks of the analytical results we applied the program FIESTA [26], which is a convenient and efficient implementation of the sector decomposition algorithm. The color factors of the individual Feynman diagrams have been computed with the program COLOR [27].

In our calculation we allowed for a general gauge parameter ξ in the gluon propagator. For individual diagrams we observe the appearance of terms up to ξ^6. We have checked that the coefficients of the ξ^n terms with $n = 2$, 3, 4, 5, and 6 are zero.

In order to present our results we decompose the three-loop coefficient in the form

$$a_3 = a_3^{(3)} n_l^3 + a_3^{(2)} n_l^2 + a_3^{(1)} n_l + a_3^{(0)}, \tag{2}$$

where n_l is the number of light quarks, and the first three coefficients on the right-hand side have been presented in Ref. [8] [see Eq. (6)]. Whereas for the fermionic contributions there are seven different color structures, in the case of $a_3^{(0)}$ there are only two. Note that the result of all color structures containing a factor C_F are generated by iterations of lower-order contributions and thus do not contribute to $a_3^{(0)}$. Diagrammatically, such contributions are easily

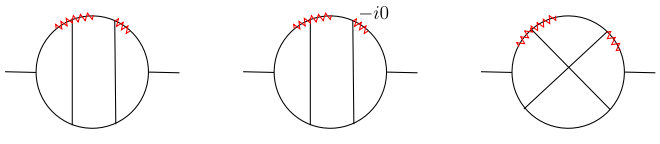

FIG. 3 (color online). Three-loop master integrals where the $\mathcal{O}(\epsilon)$ part is only known numerically. The label "$-i0$" indicates that instead of the static propagator $1/(p_0 + i0)$ there is the propagator $1/(p_0 - i0)$.

identified since the corresponding Feynman integrals contain so-called pinch contributions of the form $1/(p_0 + i0) \times 1/(p_0 - i0)$, where p_0 is the zeroth component of a loop momentum. Our result for $a_3^{(0)}$ reads

$$a_3^{(0)} = 502.24(1) C_A^3 - 136.39(12) \frac{d_F^{abcd} d_A^{abcd}}{N_A}. \tag{3}$$

Similarly to the fermionic contribution, new color invariants appear which can be traced back to Feynman diagrams as the third one in the second row of Fig. 1. Expressed in terms of N_c one has $d_F^{abcd} d_A^{abcd}/N_A = (N_c^3 + 6N_c)/48$. The coefficient of $d_F^{abcd} d_A^{abcd}$ has already been presented in Refs. [28,29], but the coefficient of the C_A^3 term is new.

Let us now discuss the numerical effect of the three-loop contribution to the static potential. Inserting the numerical results for the coefficients a_i in Eq. (1) we obtain

$$\begin{aligned} V(|\vec{q}|) = -\frac{4\pi C_F \alpha_s(|\vec{q}|)}{\vec{q}^2}\Bigg[1 &+ \frac{\alpha_s}{\pi}(2.5833 - 0.2778 n_l) \\ &+ \left(\frac{\alpha_s}{\pi}\right)^2 (28.5468 - 4.1471 n_l + 0.0772 n_l^2) \\ &+ \left(\frac{\alpha_s}{\pi}\right)^3 (209.884(1) - 51.4048 n_l + 2.9061 n_l^2 \\ &- 0.0214 n_l^3) + \cdots\Bigg], \end{aligned} \tag{4}$$

where $\mu^2 = \vec{q}^2$ has been adopted in order to suppress the infrared logarithm and the ellipses denote higher order terms in α_s. The term "209" in the three-loop coefficient receives a large contribution ("211") from the C_A^3 term whereas the new color structure only contributes with a coefficient "-2." From Eq. (4) we observe at one-, two-, and three-loop order a large screening of the nonfermionic contributions by the n_l terms which is most prominent in the case of a_3 for $n_l = 5$. Here the difference between $a_3^{(0)}$ and the fermionic contribution is one order smaller than the individual pieces.

In Table I we show the numerical evaluation of the square bracket of Eq. (4) for the charm, bottom, and top quark case, i.e., for $n_l = 3$, 4, and 5, adopting the appropriate values of α_s. For charm the three-loop corrections are almost as big as the one- and two-loop contributions, whereas for the bottom the three-loop contribution is already a factor of 4 smaller than the two-loop one. In the case of the top quark, one observes a good convergence: the three-loop term is already a factor of 10 smaller than the two-loop counterpart.

As already mentioned above, $V(|\vec{q}|)$ itself is not a physical quantity. Hence, let us consider the ground state energy E_1 of a heavy quarkonium system which has been evaluated to the third order in perturbation theory (PT) in Ref. [30], where the contribution from a_3 has been kept unevaluated. We are now in a position to complete the numerical analysis. It is convenient to write the perturbative contribution to E_1 in the form

TABLE I. Radiative corrections to the potential $V(|\vec{q}|)$ where the tree-level result is normalized to 1 [cf. Eq. (4)]. In the second column we also provide the numerical value of α_s corresponding to the soft scale where $\mu \approx m_q \alpha_s$ (m_q being the heavy-quark mass).

n_l	$\alpha_s^{(n_l)}$	One loop	Two loop	Three loop
3	0.40	0.2228	0.2723	0.1677
4	0.25	0.1172	0.08354	0.02489
5	0.15	0.05703	0.02220	0.002485

$$E_1^{\mathrm{PT}} = E_1^C + \delta E_1^{(1)} + \delta E_1^{(2)} + \delta E_1^{(3)} + \cdots, \qquad (5)$$

with the Coulomb energy $E_1^C = -C_F^2 \alpha_s^2 m_q/4$. m_q is the heavy-quark mass and the superscript in brackets indicates the order in perturbation theory. Adopting for the renormalization scale the choice $\mu_S = C_F \alpha_s(\mu_S) m_q$, we obtain

$$\begin{aligned}
\delta E_1^{(3)}|_{\mathrm{charm}} &= \alpha_s^3 E_1^C [129.79 + 5.241|_{a_3} + 15.297 \ln(\alpha_s)], \\
\delta E_1^{(3)}|_{\mathrm{bottom}} &= \alpha_s^3 E_1^C [104.82 + 3.186|_{a_3} + 15.297 \ln(\alpha_s)], \\
\delta E_1^{(3)}|_{\mathrm{top}} &= \alpha_s^3 E_1^C [83.386 + 1.473|_{a_3} + 15.297 \ln(\alpha_s)],
\end{aligned} \qquad (6)$$

where the contribution from a_3 has been marked separately. One observes that the numerical effect amounts to between 1% and 4% of the nonlogarithmic constant.

Finally, it is interesting to compare our results with the predictions obtained on the basis of certain assumptions on the perturbative expansion. In Ref. [31] a Padé approximation in the coupling constant has been performed, whereas the findings of Ref. [32] are based on renormalon studies. For $a_3^{(0)}/4^3$ they predict 313 and 292, respectively, which overshoots the exact result by 40%–50%.

More recently, a detailed comparison of the perturbative result with lattice simulations has been performed [33] with the goal of extracting $a_3^{(0)}$. After transforming the result of Ref. [33] to momentum space using the formulas provided in their appendix, one obtains $202 \leq a_3^{(0)}/4^3 \leq 337$. Thus the lower limit of the (relatively big) interval covers the exact result.

In conclusion, in this Letter the three-loop corrections to the static potential have been completed by evaluating the gluonic contribution. Our main result can be found in Eq. (3), where the three-loop coefficients are given for general color structure. Numerical sizable corrections are observed for the nonfermionic contributions which are partly canceled by the fermionic corrections evaluated in Ref. [8].

Let us stress that the static potential constitutes a fundamental quantity of QCD. It represents a building block in many physical quantities such as the determination of the bottom quark mass from the $\Upsilon(1S)$ bound state or the third-order correction to top quark threshold production cross section at a future electron positron linear collider, which would result in the most precise value for the top quark mass. The static energy is also a crucial object when comparing perturbation theory and lattice simulations (see, e.g., Refs. [33–36]). We also want to mention the extraction of the strong coupling constant from lattice simulations where again the static potential and in particular a_3 plays an important role [37,38].

We would like to thank Alexander Penin for many useful discussions and communications. We are grateful to Alexey Pak for assistance and to Johann Kühn for useful comments. This work is supported by DFG through SFB/TR 9 "Computational Particle Physics" and RFBR, Grant No. 08-02-01451.

Note added.—While finishing this Letter we became aware of the work by Anzai *et al.* [39], where $a_3^{(0)}$ has also been computed. We agree with their Eq. (10); however, we obtain a better precision.

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