Necessary Condition for Frequency Synchronization in Network Structures

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We present the necessary condition for complete frequency synchronization of phase-coupled oscillators in network structures. The surface area of a set of sites is defined as the number of links between the sites within the set and those outside the set. The necessary condition is that the surface area of any set of cN $(0 < c < 1)$ oscillators in the N-oscillator system must exceed $\sqrt{N}$ in the limit $N \to \infty$. We also provide the necessary condition for macroscopic frequency synchronization. Thus, we identify networks in which one or both of the above mentioned types of synchronization do not occur.

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All systems showing collective synchronization comprise a network that consists of elements and interactions between them [1]. Generally, synchronized dynamics depend on the structure of the network. Therefore, it is important to reveal how the structure of the network causes effects on the synchronized dynamics. Once this is revealed, it will be possible to coordinate synchronization through the construction of the appropriate network. Conversely, it will also be possible to predict the network structure by observing the synchronized dynamics [2].

The synchronized dynamics for some networks have been studied theoretically using the Kuramoto model [3], which is the most representative model of a phase-coupled oscillator. Here, a site in a given network refers to an oscillator, while links between pairs of sites are referred to as interactions. One of the most important requirements is to confirm the existence of frequency synchronization in an infinite system. Frequency synchronization is observed in global networks [3], random networks [4], scale-free networks [5], and small-world networks [4]. Frequency synchronization is also observed in two- and three-dimensional cubic lattices [6,7]. However, it has been theoretically proven that there is no synchronized solution for one-dimensional lattices [8] when the natural frequencies are independent random variables chosen according to a distribution with a finite variance.

Although it is known that some networks show synchronization, as described above, the conditions necessary for a network structure to show synchronization have not been clarified thus far. The previously reported parameters for network structure characterization cannot be used for determining the exact conditions for synchronization in the present study. Let us consider an example. In Watts-Strogatz (WS)-type small-world networks [4], a small characteristic path length helps accelerate the synchronization process, as the critical coupling strength decreases with a decrease in the characteristic path length. However, this does not imply that a small characteristic path length is the necessary condition for synchronization.

In this Letter, we discuss the necessary condition that a network structure should satisfy for frequency synchronization of general phase-coupled oscillators. We introduce a new parameter called *surface area* to describe critical network structure for synchronization. On the basis of the condition, we investigate the characteristics of various networks and identify networks in which frequency synchronization is not observed.

We consider a phase-coupled oscillator system described as

$$\frac{d\theta_i}{dt} = \omega_i + K \sum_{j=1}^{N} A_{ij} h(\theta_j - \theta_i), \quad (1)$$

where θ_i and ω_i are the phase and the natural (uncoupled) frequency of oscillator i, respectively. N represents the number of oscillators. The positive constant K denotes the coupling strength. The time variable, denoted by t, is scaled by the characteristic time, and all the variables are dimensionless. We assume that the natural frequencies are randomly chosen for a certain distribution function $g(\omega)$ with a finite variance σ^2. The interaction $h(\theta)$ is assumed to be a continuous odd periodic function. Note that $h(\theta)$ is a bounded function. We denote the structure of the interaction network by an adjacency matrix $\mathbf{A}$, where $A_{ij} = 1$ if oscillators i and j interact with each other, and $A_{ij} = 0$ otherwise. The diagonal elements are $A_{ii} = 0$. We assume that the marix is symmetric, i.e., $A_{ij} = A_{ji}$. When the interaction term is a sinusoidal function, that is, $h(\theta) = \sin(\theta)$, Eq. (1) represents the Kuramoto oscillator [3].

Coupled frequency is defined in terms of the time-averaged phase velocity as follows:

$$\left\langle \frac{d\theta_i}{dt} \right\rangle_t = \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{d\theta_i}{dt} dt. \quad (2)$$

If oscillators i and j interact with each other and satisfy the condition $\langle \frac{d\theta_i}{dt} \rangle_t = \langle \frac{d\theta_j}{dt} \rangle_t$, they are said to be mutually entrained. We define a cluster as a group of sites that are mutually entrained. The order parameter for frequency synchronization is defined by

$$r = \frac{N_s}{N}, \quad (3)$$

where N_s is the size of the largest cluster in the system. We denote the state with $r = 1$ as the complete synchronization (CS) state and that with $0 < r < 1$ as the macroscopic synchronization (MS) state for an infinite system.

We give the necessary condition for CS of network structures at a finite K. We use the concept of the Brownian bridge, which has been used to prove that there is no synchronized solution for a one-dimensional lattice [8]. First, we obtain the long-time average of the terms on both sides of Eq. (1),

$$\left\langle \frac{d\theta_i}{dt} \right\rangle_t = \omega_i + K \sum_j A_{ij} \langle h(\theta_j - \theta_i) \rangle_t. \tag{4}$$

By adding up N equations, we obtain

$$\sum_{i=1}^{N} \left\langle \frac{d\theta_i}{dt} \right\rangle_t = \sum_{i=1}^{N} \omega_i. \tag{5}$$

Because $\langle h(\theta_j - \theta_i) \rangle_t$ is an odd function and $A_{ij} = A_{ji}$, there are no interaction terms in Eq. (5). Next, we choose cN arbitrary sites from N sites, where c is a constant in the range $0 < c < 1$. Summation of the cN equations given by Eq. (4) gives

$$\sum_{i\in\{cN\}} \left\langle \frac{d\theta_i}{dt} \right\rangle_t = \sum_{i\in\{cN\}} \omega_i + K \sum_{\text{surface}(\{cN\})} \langle h(\theta_j - \theta_i) \rangle_t, \tag{6}$$

where the label $\{cN\}$ denotes the set of cN sites that we have chosen. surface($\{cN\}$) indicates the set of links between the sites in $\{cN\}$ and other sites outside $\{cN\}$; this is conceptually shown in Fig. 1. The interactions in $\{cN\}$ mutually cancel out, while those at the surface of $\{cN\}$ remain. The difference between Eq. (5) and $1/c$ times Eq. (6) is given as follows:

$$\sum_{i=1}^{N} \left\langle \frac{d\theta_i}{dt} \right\rangle_t - \frac{1}{c} \sum_{i\in\{cN\}} \left\langle \frac{d\theta_i}{dt} \right\rangle_t = \left\{ \sum_{i=1}^{N} \omega_i - \frac{1}{c} \sum_{i\in\{cN\}} \omega_i \right\} - \frac{1}{c} K \sum_{\text{surface}(\{cN\})} \langle h(\theta_j - \theta_i) \rangle_t. \tag{7}$$

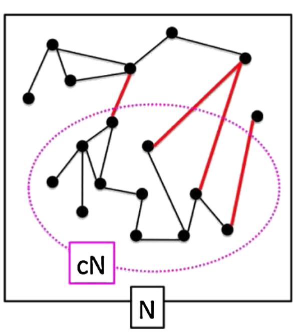

FIG. 1 (color online). Definition of surface and surface area. cN $(0 < c < 1)$ sites are chosen from N sites in a network. The surface of a set $\{cN\}$ is defined as the set of links between the sites within $\{cN\}$ and those outside $\{cN\}$. The surface area is defined as the number of links at the surface.

When all oscillators are assumed to have the same frequency $\langle \frac{d\theta_i}{dt} \rangle_t = \Omega$, the left-hand side (LHS) of Eq. (7) reduces to zero:

$$0 = \left\{ \sum_{i=1}^{N} \omega_i - \frac{1}{c} \sum_{i\in\{cN\}} \omega_i \right\} - \frac{1}{c} K \sum_{\text{surface}(\{cN\})} \langle h(\theta_j - \theta_i) \rangle_t. \tag{8}$$

Note that Eq. (8) is true for any set of $\{cN\}$ when CS occurs. The term $\{\sum_{i=1}^{N} \omega_i - \frac{1}{c} \sum_{i\in\{cN\}} \omega_i\} \equiv X(\{\omega_i\})$ is a random variable determined from the natural frequencies. The exact variance of $X(\{\omega_i\})$ given by $(\frac{1}{c} - 1)\sigma^2 N$ can be calculated. The standard deviation of $X(\{\omega_i\})$, which represents a typical value of $X(\{\omega_i\})$, is of the order of $\sqrt{N}$. For Eq. (8) to have a solution for a finite K in the limit $N \to \infty$, the summation of the interaction terms must be of the order of $\sqrt{N}$. Because h is a bounded function, the number of links in surface($\{cN\}$), which is denoted by $S(\{cN\})$, must be greater than $\sqrt{N}$. We call $S(\{cN\})$ the *surface area* of the set $\{cN\}$. Thus, the necessary condition for CS [9] is given by

$$\lim_{N\to\infty} \frac{S(\{cN\})}{\sqrt{N}} > 0 \quad \text{for any set } \{cN\}. \tag{9}$$

Here, we can give a physical explanation for the necessary condition for synchronization as follows: The LHS of Eq. (7) represents the difference between the average coupled frequency over the entire network and that in the set $\{cN\}$. The term $X(\{\omega_i\})$ represents the difference between the average natural frequency over the entire network and that in the set $\{cN\}$. The interactions in surface$\{cN\}$ compensate for this difference and reduce the aforementioned difference between the average coupled frequencies. Consequently, for the entire system to be synchronized, the surface area of any set $\{cN\}$ must be greater than $\sqrt{N}$.

The exact necessary condition for CS is given by Eq. (9). To confirm whether the system satisfies the necessary condition, we must search for a set $\{cN\}$ that has the minimum surface area. Let us investigate various networks on the basis of this condition, as shown in Fig. 2. The minimum surface area ($S_{\min}$) of the one-dimensional lattice is 2. Hence, CS is not achieved in the one-dimensional lattice. $S_{\min}$ for the two-dimensional lattice is $2^2\sqrt{cN}$, and the necessary condition is satisfied. We employ the WS model [10] as an example of a small-world network. The WS model can be described as follows: We first design a one-dimensional ring and connect each site to the first through the kth neighbors, where k is a positive integer. Next, with a probability p, we randomly rewire each link. The small-world network is realized for a finite p, which is close to zero. Here, we choose cN sites that are mutually neighboring along the one-dimensional ring in order to minimize the surface area of a set consisting of cN sites. The surface area is approximately expressed in terms of the number of rewirings, which consist of outgoing and in-

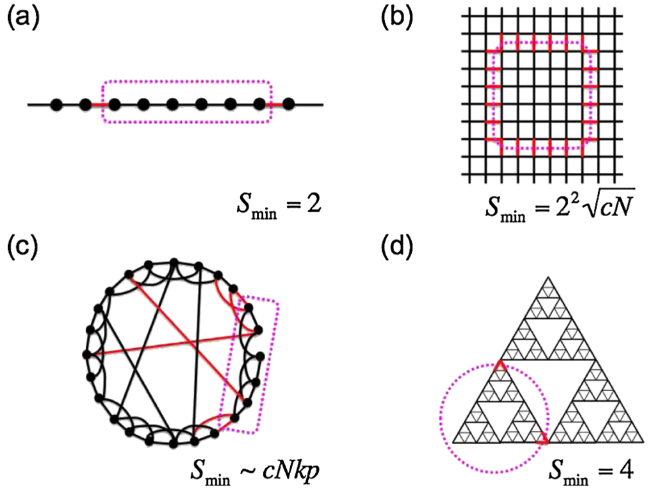

FIG. 2 (color online). Minimum surface area $S_{\min}$ for various networks. $S_{\min}$ is (a) 2 in a one-dimensinal lattice, (b) $2^2\sqrt{cN}$ in a square lattice, (c) $cNkp$ in a WS-type small-world network, and (d) 4 in the Sierpinski gasket.

coming wires, as follows: $S(\{cN\}) = cNkp(1-c) + (1-c)Nkpc = 2c(1-c)Nkp$. We choose a small c to minimize $S(\{cN\})$ and to ensure that $S_{\min} \sim cNkp$. The results show that the WS small-world network satisfies the necessary condition for CS. In the Sierpinski gasket, $S_{\min}$ is 4, and hence, CS is not achieved. Thus, the proposed necessary condition can be effectively used to identify network systems in which CS does not occur.

We also derive the necessary condition for MS, for which the order parameter r satisfies the condition $0 < r < 1$, similar to the case for CS. To discuss nontrivial MS, we append the assumption that $g(\omega)$ has no delta-function singularity. First, we add up rN equations given by Eq. (4) and obtain the following equation:

$$\sum_{i\in\{rN\}} \left\langle \frac{d\theta_i}{dt} \right\rangle_t = \sum_{i\in\{rN\}} \omega_i + K \sum_{\mathrm{surface}(\{rN\})} \langle h(\theta_j - \theta_i)\rangle_t. \quad (10)$$

Next, we choose cN $(0 < c < r)$ arbitrary sites from the rN sites. Through the summation of cN equations, which are part of the rN equations, we obtain

$$\sum_{i\in\{cN\}} \left\langle \frac{d\theta_i}{dt} \right\rangle_t = \sum_{i\in\{cN\}} \omega_i + K \sum_{\mathrm{surface}(\{cN\})} \langle h(\theta_j - \theta_i)\rangle_t. \quad (11)$$

From Eq. (10) and (11), we obtain

$$\frac{1}{r}\sum_{i\in\{rN\}} \left\langle \frac{d\theta_i}{dt} \right\rangle_t - \frac{1}{c}\sum_{i\in\{cN\}} \left\langle \frac{d\theta_i}{dt} \right\rangle_t = \left\{\frac{1}{r}\sum_{i\in\{rN\}} \omega_i - \frac{1}{c}\sum_{i\in\{cN\}} \omega_i\right\} + K\left\{\frac{1}{r}\sum_{\mathrm{surface}(\{rN\})} \langle h(\theta_j - \theta_i)\rangle_t - \frac{1}{c}\sum_{\mathrm{surface}(\{cN\})} \langle h(\theta_j - \theta_i)\rangle_t\right\}. \quad (12)$$

When all the oscillators in $\{rN\}$ are assumed to have the same frequency $\langle\frac{d\theta_i}{dt}\rangle_t = \Omega$, the LHS of Eq. (12) reduces to zero. Under this assumption, Eq. (12) can be rewritten as follows:

$$\begin{aligned} 0 = &\left\{\frac{1}{r}\sum_{i\in\{rN\}} \omega_i - \frac{1}{c}\sum_{i\in\{cN\}} \omega_i\right\} \\ &- \frac{1}{c}K \sum_{\mathrm{surface}(\{cN\})j\in\{rN\}} \langle h(\theta_j - \theta_i)\rangle_t \\ &+ K\left\{-\frac{1}{c}\sum_{\mathrm{surface}(\{cN\})j\notin\{rN\}} \langle h(\theta_j - \theta_i)\rangle_t \right. \\ &\left. + \frac{1}{r}\sum_{\mathrm{surface}(\{rN\})} \langle h(\theta_j - \theta_i)\rangle_t\right\}. \end{aligned} \quad (13)$$

Note that we classify the links in surface($\{cN\}$) into two groups as follows: (i) surface$(\{cN\})_{j\in\{rN\}}$, where a site i in $\{cN\}$ is connected with a site j outside $\{cN\}$ and within $\{rN\}$. (ii) surface$(\{cN\})_{j\notin\{rN\}}$, where a site i within $\{cN\}$ is connected with a site j outside $\{rN\}$. Equation (13) holds for any set $\{cN\}$ when synchronization occurs in $\{rN\}$. Here, we evaluate a typical value of $\{\frac{1}{r}\sum_{i\in\{rN\}}\omega_i - \frac{1}{c}\sum_{i\in\{cN\}}\omega_i\} \equiv X_r(\{\omega_i\})$. When the rN equations are chosen independently of ω_i, the exact standard deviation of $X_r(\{\omega_i\})$ is given by $\sqrt{(\frac{1}{c}-\frac{1}{r})\sigma^2 N}$. Even when the rN equations are chosen from the equations of which ω_i is in a limited range, the standard deviation is of the order of $\sqrt{N}$. There is no set $\{rN\}$ that has the smaller typical value of $X_r(\{\omega_i\})$ than $\sqrt{N}$. Therefore, for Eq. (13) to have a solution in the limit $N \to \infty$, the summation of the interactions must be of the order of $\sqrt{N}$.

Let us recollect the interactions that help reduce the difference between the average coupled frequencies in $\{rN\}$ and $\{cN\}$. It is apparent that the interactions in surface$(\{cN\})_{j\in\{rN\}}$ are suitable for this purpose. Thus, it is concluded that surface$(\{cN\})_{j\in\{rN\}}$ mainly contributes to reducing the LHS of Eq. (12) to zero. It is impossible for synchronization to occur in $\{rN\}$ when the summation of surface$(\{cN\})_{j\in\{rN\}}$ is not of the order of $\sqrt{N}$. Thus, the necessary condition for MS is given by

$$\lim_{N\to\infty} \frac{S(\{cN\})_{j\in\{rN\}}}{\sqrt{N}} > 0 \quad \text{for any } \{cN\} \quad (14)$$

in the cluster $\{rN\}$, where $S(\{cN\})_{j\in\{rN\}}$ represents the number of links in surface$(\{cN\})_{j\in\{rN\}}$.

Now, we discuss the information provided by the necessary condition for MS. On the basis of the proposed condition, we can identify networks in which MS does not occur. If there is no set $\{rN\}$ that satisfies the necessary condition for a network, the network does not show MS. Let us investigate various examples. In the case of the one-dimensional lattice and the Sierpinski gasket, no set $\{rN\}$ that satisfies the necessary condition is found. Hence, neither CS nor MS occurs in these networks. On the other hand, in random networks and small-world networks, we can easily find a set $\{rN\}$ that satisfies the necessary

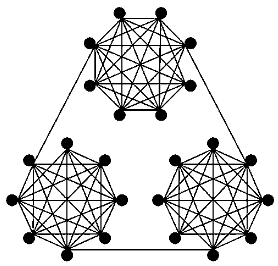

FIG. 3. Schematic of the connected caveman graph consisting of $N/(k+1)$ complete graphs or *caves*. A link in each *cave* is rewired to connect the cave with its neighbor.

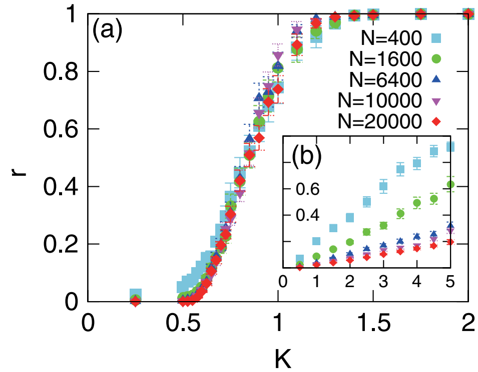

FIG. 4 (color online). The order parameter r for the Kuramoto-oscillator system on our network model for (a) $N^{0.5} < k < N^{0.6}$ and (b) $N^{0.2} < k < N^{0.3}$.

condition. These results are consistent with those reported in previous studies [4,6,8].

Here, we present an example in which the necessary condition for MS is satisfied, while that for CS is not. We employ the connected caveman graph [11] as an example. The connected caveman graph is obtained as follows: First, $N/(k+1)$ complete graphs or *caves* are prepared. Each cave consists of $k+1$ sites. A link in each cave is rewired to connect the cave with its neighbor, as shown in Fig. 3. It is assumed that k satisfies the condition $k \sim N$. If a set $\{rN\}$ is assumed to be a cave, the necessary condition for MS is satisfied, as the surface of any set $\{cN\}$ is of the order of N. On the other hand, CS is impossible, as the surface area of each cave is 2. Thus, the connected caveman graph shows only MS. When k satisfies the condition $k \sim N^{\alpha}$ $(0 \leq \alpha < 1)$, neither CS nor MS occurs; this is because we cannot find the set $\{rN\}$ that satisfies the necessary condition for MS.

Finally, we investigate the sufficiency of the proposed condition for MS numerically. We introduce a network model, in which the minimum surface area can be controlled simply. The definition is explained as follows: First, we prepare a one-dimensional ring, in which the nearest-neighbor pairs are all connected to each other. Next, we randomly add N links that connect kth-neighbor pairs, where k is randomly chosen from a uniform distribution on $N^{\beta} < k < N^{\beta'}$ $(0 < \beta < \beta' \leq 1)$. β and β' are parameters to control the minimum surface area. A set of cN sites that are mutually neighboring along the ring has the minimum surface area, which satisfies $N^{\beta} \leq S(\{cN\}) \leq N^{\beta'}$. We simulate the synchronized dynamics of the Kuramoto oscillators on this network when $g(\omega)$ is assumed to be the Gaussian distribution centered about 0 with unit variance. Figure 4 shows the behavior of r for (a) $N^{0.5} < k < N^{0.6}$ and (b) $N^{0.2} < k < N^{0.3}$. While MS disappears gradually as N is increased for the case (b), MS transition occurs even when N is increased for the case (a). We numerically confirm that MS transition occurs for $\beta' > \beta \geq 0.5$. The details will be presented in a forthcoming paper [12]. This result indicates a possibility that the proposed condition is both necessary and sufficient.

In this Letter, the necessary conditions for CS and MS are provided. At present, there are no reported examples of networks that satisfy the proposed condition but do not show synchronization. It is an open problem whether the proposed conditions are exactly necessary and sufficient.

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