Quantum Metrology with Two-Mode Squeezed Vacuum: Parity Detection Beats the Heisenberg Limit

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We study the sensitivity and resolution of phase measurement in a Mach-Zehnder interferometer with two-mode squeezed vacuum ($\bar{n}$ photons on average). We show that superresolution and sub-Heisenberg sensitivity is obtained with parity detection. In particular, in our setup, dependence of the signal on the phase evolves $\bar{n}$ times faster than in traditional schemes, and uncertainty in the phase estimation is better than $1/\bar{n}$, and we saturate the quantum Cramer-Rao bound.

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Different physical mechanisms contribute to phase measurement. Therefore, improved phase estimation benefits multiple areas of scientific research, such as quantum metrology, imaging, sensing, and information processing. As a consequence, enormous efforts have been devoted to improve the resolution and sensitivity of interferometers.

In what follows, we direct our attention to quantum interferometry. The benchmark that quantum interferometry is compared against is one with coherent light input and intensity difference measurement at the output of a Mach-Zehnder interferometer (MZI). Without nonlinear interaction between photons in the MZI, phase sensitivity of this benchmark is shot-noise limited (SNL), namely $\Delta\varphi_{\rm SNL} = 1/\sqrt{\bar{n}}$, where $\bar{n}$ is the average number of photons [1].

In 1981, Caves pointed out that by using coherent light together with squeezed vacuum one could beat SNL $\Delta\varphi < \Delta\varphi_{\rm SNL}$ (supersensitivity) [2]. In the work of Boto *et al.*, it was demonstrated that by exploiting special states of light, such as $N00N$ states (N-particle path-entangled states $|N, 0\rangle + |0, N\rangle$), it is possible to beat the Rayleigh diffraction limit in imaging and lithography (superresolution), while also beating SNL [3–6]. Finally, it was shown in Ref. [7] that input state entanglement is important in order to achieve supersensitivity.

We could place a limit on the supersensitivity, if we assume the validity of the Heisenberg uncertainty principle for phase and photon number uncertainties: $\Delta n\Delta\varphi \geq 1$. This relationship easily translates into the Heisenberg limit on the phase sensitivity of a N-photon state, $\Delta\varphi_{\rm HL} = 1/N$, due to the bound on photon number difference, $\Delta n \leq N$. In order to define a similar limit for states with an indefinite number of photons, characterized by the mean value $\bar{n}$, an argument about finite energy is given—thus imposing the following bound $\Delta n \leq \bar{n}$. Such a notion about the Heisenberg limit can be traced back to, for example, work by Ou [8], where he speculates that the fundamental limit set by quantum mechanics on sensitivity is the Heisenberg limit, $\Delta\varphi_{\rm HL} = 1/\bar{n}$, since all analyses up until then had not shown better than $1/\bar{n}$ sensitivity.

Experimental realization of a supersensitive phase measurement that would be better than a SNL measurement with coherent light have been hindered by the fact that entangled states of light, with large number of photons, are difficult to obtain. Therefore we turn our attention to the brightest (experimentally available) nonclassical light—-two-mode squeezed vacuum (TMSV). A state of TMSV is a superposition of twin Fock states $|\psi_{\bar{n}}\rangle = \sum_{n=0}^{\infty}\sqrt{p_n(\bar{n})}\,|n, n\rangle$, where the probability of a twin Fock state $|n, n\rangle = |n\rangle_A|n\rangle_B$ to be present depends on the average number of photons in both modes of TMSV, $\bar{n}$, in the following way, $p_n(\bar{n}) = (1 - t_{\bar{n}})t_{\bar{n}}^n$ with $t_{\bar{n}} = 1/(1 + 2/\bar{n})$ [9].

Light entering a MZI in the TMSV state exits a lossless interferometer in the state $|\psi_f\rangle = \hat{U}_{\rm MZI}|\psi_{\bar{n}}\rangle$, where the MZI is described by the unitary transformation $\hat{U}_{\rm MZI}$ (Fig. 1). This transformation, in terms of the field operators for the optical modes $\hat{a}$ and $\hat{b}$, is $\hat{U}_{\rm MZI} = \hat{U}\hat{P}_\varphi\hat{U} = \exp[\varphi(\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a})/2]$, where $\hat{P}_\varphi = \exp(-i\varphi\hat{G})$ describes accumulation of a phase difference φ; and $\hat{U} = \exp[i\frac{\pi}{4}(\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger)]$ describes the 50-50 beam splitter, with a $\pi/2$ phase shift for the reflected mode. In a linear medium the generator of phase evolution is $\hat{G} = (\hat{n}_A - \hat{n}_B)/2$, where $\hat{n}_A = \hat{a}^\dagger\hat{a}$ and $\hat{n}_B = \hat{b}^\dagger\hat{b}$ are the photon number operators in each mode.

Phase estimation is based on the detection of light at the outputs of a MZI. Not all detection schemes are capable of exploiting the full potential of nonclassical light to be supersensitive and superresolving. For example, intensity difference measurement, which is standard for optical interferometry with coherent light, is not phase sensitive at all if TMSV input is used [10]. In our Letter, we consider parity detection for our measuring scheme. The parity operator on an output mode A is $\hat{\Pi}_A = \exp(i\pi\hat{n}_A)$. Parity

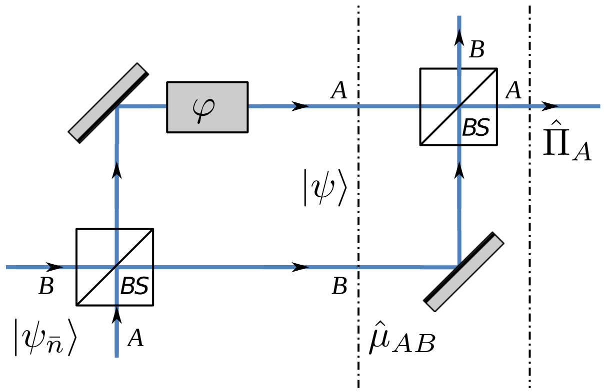

FIG. 1 (color online). The Mach-Zehnder interferometer used in the calculations. The two-mode squeezed vacuum input state $|\psi_{\bar{n}}\rangle$ is indicated together with the intermediate state $|\psi\rangle = \hat{P}_\varphi \hat{U}|\psi_{\bar{n}}\rangle$. Vertical dash-dotted lines indicate places where two measurements $\hat{\mu}_{AB}$ and $\hat{\Pi}_A$ are to be implemented.

was originally discussed in the context of trapped ions by Bollinger *et al.* [11] and later adopted for optical interferometry by Gerry and Campos [12,13]. Supersensitivity with this detection strategy has been shown for several classes of input states [14]. Finally, parity detection was also shown to allow better than classical resolution with coherent light while keeping SNL phase sensitivity [15].

A parity measurement on mode A at the output of the MZI is computed by $\langle\hat{\Pi}_A\rangle = \langle\psi_f|\hat{\Pi}_A|\psi_f\rangle$. It has been shown in Ref. [15] that the parity measurement on mode A after the final beam splitter is equivalent to the measurement that is constructed from all the $|M, M'\rangle \rightarrow |M', M\rangle$ projectors as follows [15]

$$\hat{\mu}_{AB} = \sum_{N=0}^{\infty}\sum_{M=0}^{N} |N-M, M\rangle\langle M, N-M|, \qquad (1)$$

acting on the inner modes of the MZI, such that $|\psi\rangle = \hat{P}_\varphi \hat{U}|\psi_{\bar{n}}\rangle$. Our use of the $\hat{\mu}_{AB}$ operator here highlights the fact that parity detection combined with a 50-50 beam splitter provides a measurement scheme that includes all of the phase-carrying off-diagonal terms in the two-mode density matrix [15]. Calculation of $\langle\hat{\mu}_{AB}\rangle$ simplifies significantly once it is noted that such an operator, as well as a beam splitter, does not change the total number of photons in the state. Thus a lossless MZI with the parity detection scheme does not mix twin Fock states with different number of photons giving:

$$\langle\hat{\Pi}_A\rangle = (1 - t_{\bar{n}}) \sum_{n=0}^{\infty} t_{\bar{n}}^n \langle\hat{\Pi}_A\rangle_n, \qquad (2)$$

where $\langle\hat{\Pi}_A\rangle_n = \langle n, n|\hat{U}_{\rm MZI}^\dagger \hat{\Pi}_A \hat{U}_{\rm MZI}|n, n\rangle$ is the expectation value of the parity operator for the twin Fock state input. In turn, the expression $\langle\hat{\Pi}_A\rangle_n = (-1)^n P_n[\cos(2\varphi)]$, given in terms of Legendre polynomials P_n, could be found in Ref. [16]. Finally, one can identify our expression in Eq. (2) with the generating function for Legendre polynomials [17] and arrive to the following:

$$\langle\hat{\Pi}_A\rangle_{\varphi+\pi/2} = \langle\hat{\mu}_{AB}\rangle_\varphi = \frac{1}{\sqrt{1 + \bar{n}(\bar{n}+2)\sin^2\varphi}}, \qquad (3)$$

where an additional $\pi/2$ phase shift was introduced. Equation (3) is the central result of this Letter and, in what follows, it will be used to study the resolution and sensitivity of our proposed scheme.

Let us compare here the signal outcomes of the TMSV scheme with $\bar{n} = 10$ to coherent-state-based optical interferometry with $\bar{n} = 100$ (see Fig. 2). Intensity difference measurement, with a coherent state at the input of the MZI, exhibits classical interference—a sinusoidal dependence on the phase with an intensity independent period of 2π. In the case of parity detection with the coherent state input, it was shown in Ref. [15] that $\langle\hat{\Pi}_A\rangle = \exp[-2\bar{n}\sin^2(\varphi/2)]$ with a 2π period and a feature at the phase origin that is narrower than the classical curve by a factor of $\delta\varphi = 1/\sqrt{\bar{n}}$. In our case, the width of the feature is further reduced by $\sqrt{\bar{n}+2}$ times. Therefore, the peak in Fig. 2 is as narrow for a $\bar{n} = 10$ TMSV as for a $\bar{n} = 100$ coherent state.

The other aspect of optical interferometry is its phase sensitivity that is quantified by an average value of how much the measured phase could differ, $\Delta\varphi$, from the actual value. Therefore the most sensitive measurement will have the smallest $\Delta\varphi$. The smallest value, however, could not be smaller than $\Delta\varphi_{\min} = 1/\sqrt{F_Q}$ [18], where the quantum Fisher information F_Q for a pure state is just four times the variance $\Delta\hat{G}^2$ of the phase evolution generator given above. This results in a measurement independent limit of

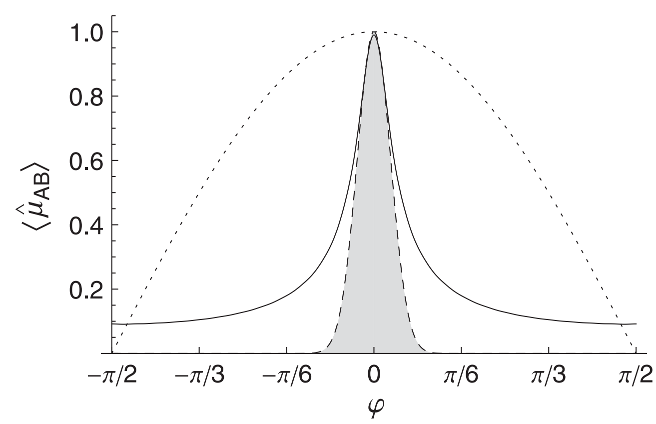

FIG. 2. Measured signals at the output of the MZI with coherent light ($\bar{n} = 100$) and TMSV ($\bar{n} = 10$) inputs against accumulated phase difference. Solid and dashed lines are the outputs of the parity measurement for TMSV and coherent light, respectively. The dotted line, given for comparison, is a scaled down output of intensity difference measurement on the output of the MZI fed with coherent light. TMSV with much smaller photon number performs as well as coherent light.

the phase sensitivity of optical interferometry with a given state:

$$\Delta\varphi_{\min}^2 = \frac{1}{4\Delta\hat{G}^2}. \tag{4}$$

Our analysis shows that Eq. (4), in the case of coherent light, limits the attainable sensitivity to $\Delta\varphi_{\min}^2 = \bar{n}^{-1}$, shot noise, while for TMSV it sets a much lower limit $\Delta\varphi_{\min}^2 = (\bar{n}^2 + 2\bar{n})^{-1} < \bar{n}^{-2}$. This means that TMSV has a potential for supersensitive phase estimation with phase sensitivity better than the Heisenberg limit defined as $1/\bar{n}$. Let us show that phase measurement based on the parity detection discussed here can actually reach this limit.

The variance of the phase estimation based on the outcome of the parity measurement could be estimated as

$$\Delta\varphi^2 = \frac{1 - \langle\hat{\mu}_{AB}\rangle^2}{(\partial\langle\hat{\mu}_{AB}\rangle/\partial\varphi)^2}, \tag{5}$$

which is a ratio of detection noise to the rate at which signal changes as a function of phase. We have shown that the rate of the signal change is much higher than in the case of coherent state input. Thus, if the parity measurement on the squeezed vacuum is no noisier than on the coherent state, sensitivity improvement is expected.

The sensitivity of the phase estimation in our scheme can be estimated based on Eq. (5) combined with Eq. (3):

$$\Delta\varphi = \frac{1 + \bar{n}(\bar{n}+2)\sin^2\varphi}{|\cos\varphi|\sqrt{\bar{n}(\bar{n}+2)}}, \tag{6}$$

which is presented in Fig. 3 for the case of $\varphi = 0$. One can clearly see that phase sensitivity obtained by the parity measurement at $\varphi = 0$ (the dashed line) saturates the lower bound defined by the quantum Fisher information (black dots). The expression in Eq. (6) gives dependence of the phase sensitivity of our scheme on the actual phase difference as well:

$$\Delta\varphi \approx \frac{1}{\sqrt{\bar{n}(\bar{n}+2)}}\left[1 + (2\bar{n}(\bar{n}+2)+1)\frac{\varphi^2}{2}\right], \tag{7}$$

where expansion near the phase origin was made. This dependence is, in turn, compared to the one for coherent state input, which has the following functional dependence on the phase in the vicinity of the phase origin

$$\Delta\varphi \approx \frac{1}{\sqrt{\bar{n}}}\left[1 + (2\bar{n}+1)\frac{\varphi^2}{8}\right]. \tag{8}$$

Dependence of the phase sensitivity in both cases is presented in the inset of Fig. 3 for $\bar{n} = 5$ and $\bar{n} = 25$ TMSV and for $\bar{n} = 25$ coherent state inputs. Comparison shows that our scheme has superior sensitivity in the vicinity of the phase origin but degrades rapidly as the actual phase difference deviates from zero. However, there is still a vicinity around $\varphi = 0$ where the phase sensitivity demonstrated by our scheme is better than the Heisenberg limit of $\Delta\varphi_{\mathrm{HL}} = 1/\bar{n}$. Hence, our proposed scheme for phase estimation is the first to beat the Heisenberg limit, in the absence of nonlinear interactions [19].

Beating the Heisenberg limit, defined in terms of the mean value, has demonstrated once again the importance of photon number fluctuations for phase estimation. In order to better account for photon number fluctuations, Hofmann in Ref. [20] suggested a more direct definition of the ultimate quantum limit of phase sensitivity $\Delta\varphi^2 \geq 1/\langle\hat{n}^2\rangle$, where $\langle\hat{n}^2\rangle$ indicates averaging over the squared photon numbers. Thus, in the case of high photon number fluctuations, $\Delta n^2 = \langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2 > 0$, the Hofmann limit allows for better sensitivity of the phase measurement than the Heisenberg limit. Clearly $\langle\hat{n}^2\rangle$ contains direct information about fluctuations where $\langle\hat{n}\rangle^2$ does not.

In the case of TMSV, $\Delta n^2 = \bar{n}^2 + 2\bar{n}$, and thus the sensitivity of the phase estimation is better than $1/\bar{n}$, although marginally, but it is never better than $1/\sqrt{2\bar{n}^2 + 2\bar{n}}$, which is the Hofmann limit. It is also never below the quantum Cramer-Rao lower bound set by the quantum Fisher information of the state.

In order to demonstrate that the maximal phase sensitivity could be underestimated by the Heisenberg limit if photon number fluctuations are neglected, consider the following state $\hat{\rho}(n,\theta) = \sin^2\theta|0,0\rangle\langle 0,0| + \cos^2\theta|n,n\rangle\langle n,n|$, which has $\bar{n} = 2n\cos^2\theta$. This state could appear in the context of a probabilistic twin-Fock state generation, with parity detection on a single output mode, since such a detection would not distinguish vacuum contribution from the twin-Fock contribution when all photons were routed out in the other port.

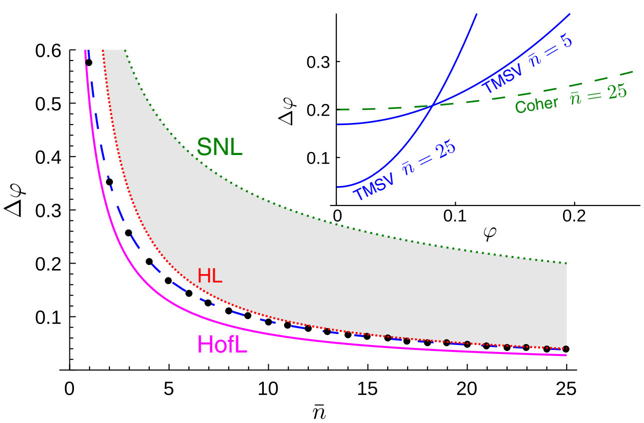

FIG. 3 (color online). Sensitivity of phase estimation obtained with the parity measurement at $\varphi = 0$ (dashed) against average total photon number. Dots are sensitivity estimation based on quantum Fisher information for integer values of $\bar{n}$. Shaded area is between dotted lines $1/\bar{n}$ and $1/\sqrt{\bar{n}}$. The solid line is for the Hofmann limit discussed in the text. Inset: Sensitivity against actual values of accumulated phase difference. Solid lines for TMSV with $\bar{n} = 5$ and $\bar{n} = 25$; dashed line for coherent light with $\bar{n} = 25$.

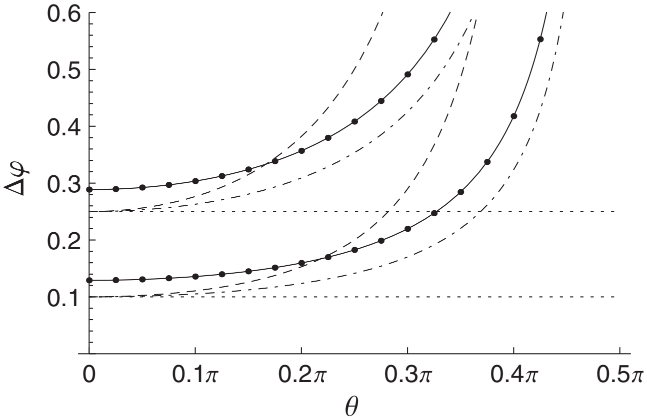

FIG. 4. Phase estimation sensitivity $\Delta\varphi$ for the state $\hat{\rho}(n,\theta)$ with $n = 2$ (top four) and $n = 5$ (bottom four) against θ. Solid lines represent sensitivity with the parity measurement which saturates the limit set by the quantum Fisher information (dots for a few selected values of θ). Dashed lines represent HL sensitivity estimation based on the averaged total photon number: $1/\bar{n}$. Dot-dashed lines give Hofmann's estimation of sensitivity discussed in the text.

Based on the quantum Fisher information, this state is capable of providing sensitivity $\Delta\varphi^2 = 1/[2n(n+1)\cos^2\theta]$, which is obtainable by the parity measurement. This dependence of the phase sensitivity is presented in Fig. 4 for $n = 2$ and $n = 5$, where the presence of the vacuum, $\theta > 0$, degrades the sensitivity but allows for $1/\bar{n}$ Heisenberg limit to be beat. However, the Hofmann limit $\Delta\varphi^2 = 1/(4n^2\cos^2\theta)$ tracks the phase sensitivity well; without being beaten.

There does exist another limit based on the highest number of photons in the state—$1/N$, with $N = 2n$ for the state considered here. However, it is not as useful as the Hofmann limit for a number of reasons: (a) it overestimates the sensitivity as it does for $\hat{\rho}(n,\theta)$; (b) information about N is not readily available in experiments; (c) for states, such as coherent and squeezed vacuum, $N = \infty$.

Finally, implementation of parity detection needs to be discussed. In the proof of principle experiments, a highly efficient photon number-resolving detector could be used. Such detectors with 95% efficiency and number-resolving capabilities in the tens of photons have been demonstrated [21–24]. However, for more practical applications, knowledge about exact photon numbers is excessive. We conjecture that a scheme, which does not require photon counting, exists, perhaps through the exploitation of optical nonlinearities [25], or projective measurements, and this is an area of ongoing research.

In conclusion, the main result of this Letter is our demonstration that optical interferometry with two-mode squeezed vacuum and parity detection provides an interferometric metrology strategy with sensitivity $\Delta\varphi < \bar{n}^{-1}$—saturating the quantum Cramer-Rao bound—and resolution $\bar{n}^{-1}$ times better than the resolution of standard (classical) interference.

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[1] S. Boixo *et al.*, Phys. Rev. Lett. **101**, 040403 (2008).
[2] C. M. Caves, Phys. Rev. D **23**, 1693 (1981).
[3] A. N. Boto *et al.*, Phys. Rev. Lett. **85**, 2733 (2000).
[4] P. Kok *et al.*, Phys. Rev. A **63**, 063407 (2001).
[5] H. Lee, P. Kok, C. Williams, and J. Dowling, J. Opt. B **6**, S796 (2004).
[6] J. P. Dowling, Contemp. Phys. **49**, 125 (2008).
[7] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. Lett. **96**, 010401 (2006).
[8] Z. Y. Ou, Phys. Rev. A **55**, 2598 (1997).
[9] C. C. Gerry and P. L. Knight, *Introductory Quantum Optics* (Cambridge University Press, Cambridge, U.K., 2005).
[10] T. Kim, O. Pfister, M. Holland, J. Noh, and J. Hall, Phys. Rev. A **57**, 4004 (1998).
[11] J. J. Bollinger, W. M. Itano, D. J. Wineland, and D. J. Heinzen, Phys. Rev. A **54**, R4649 (1996).
[12] C. C. Gerry, Phys. Rev. A **61**, 043811 (2000).
[13] C. C. Gerry and R. A. Campos, Phys. Rev. A **64**, 063814 (2001).
[14] A. Chiruvelli and H. Lee, arXiv:0901.4395v1.
[15] Y. Gao, C. F. Wildfeuer, P. M. Anisimov, H. Lee, and J. P. Dowling, arXiv:0907.2382v2.
[16] R. A. Campos, C. C. Gerry, and A. Benmoussa, Phys. Rev. A **68**, 023810 (2003).
[17] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun, National Bureau of Standards Applied Mathematics Series No. 55 (U.S. GPO, Washington, DC, 1964), p. 783.
[18] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **72**, 3439 (1994).
[19] The paper by Yurke *et al.* on SU(1,1) interferometers [B. Yurke, S. L. McCall, and J. R. Klauder, Phys. Rev. A **33**, 4033 (1986)], could have been an exception if it was not for the wrong sign in Eq. (9.31) and not mentioning better than $1/\bar{n}$ phase estimation.
[20] H. F. Hofmann, Phys. Rev. A **79**, 033822 (2009).
[21] A. J. Miller, S. W. Nam, J. M. Martinis, and A. V. Sergienko, Appl. Phys. Lett. **83**, 791 (2003).
[22] A. E. Lita, A. J. Miller, and S. W. Nam, Opt. Express **16**, 3032 (2008).
[23] C. F. Wildfeuer *et al.*, Phys. Rev. A **80**, 043822 (2009).
[24] G. Khoury, H. S. Eisenberg, E. J. S. Fonseca, and D. Bouwmeester, Phys. Rev. Lett. **96**, 203601 (2006).
[25] C. C. Gerry, A. Benmoussa, and R. A. Campos, Phys. Rev. A **72**, 053818 (2005).