# Fractional Quantum Hall State at $\nu = \frac{5}{2}$ and the Moore-Read Pfaffian

M. Storni,[1] R. H. Morf,[1] and S. Das Sarma[2]

[1]Condensed Matter Theory, Paul Scherrer Institute, CH-5232 Villigen, Switzerland

[2]Condensed Matter Theory Center, Department of Physics, University of Maryland, College Park, Maryland 20742, USA

(Received 20 December 2008; published 18 February 2010)

Using exact diagonalization we show that the spin-polarized Coulomb ground state at $\nu = \frac{5}{2}$ is adiabatically connected with the Moore-Read wave function for systems with up to 18 electrons on the surface of a sphere. The ground state is protected by a large gap for all system sizes studied. Furthermore, varying the Haldane pseudopotentials $v_1$ and $v_3$, keeping all others at their value for the Coulomb interaction, energy gap and overlap between ground- and Moore-Read state form hills whose positions and extent in the $(v_1, v_3)$ plane coincide. We conclude that the physics of the Coulomb ground state at $\nu = \frac{5}{2}$ is captured by the Moore-Read state. Such an adiabatic connection is not found at $\nu = \frac{1}{2}$, unless the width of the interface wave function or Landau level mixing effects are large enough. Yet, a Moore-Read-phase at $\nu = \frac{1}{2}$ appears unlikely in the thermodynamic limit.

DOI: 10.1103/PhysRevLett.104.076803 PACS numbers: 73.43.Cd, 71.10.Pm

One of the most intriguing strongly correlated electronic states discovered in nature is the even-denominator fractional quantum Hall effect (FQHE) at the Landau level filling factor $\nu = \frac{5}{2} = 2 + \frac{1}{2}$ [1], i.e., at the half-filled second orbital Landau level (LL) of a 2D electron system. The $\frac{5}{2}$ FQHE cannot be understood within the canonical hierarchical (Laughlin) theory, since the odd-denominator rule is a necessity to preserve the Pauli principle. A particularly interesting proposal by Moore and Read (MR) [2] extending Laughlin's ideas to quantum Hall states at half filling is the "Pfaffian" wave function (WF), characterized by quasiparticle excitations obeying non-Abelian braiding statistics [3].

The first numerical study of this WF was carried out by Greiter *et al.* [4] who considered it as a candidate for the observed FQHE at both $\nu = \frac{1}{2}$ and $\frac{5}{2}$. Their calculations done for systems on the sphere with $N_{\rm el} \leq 10$ electrons did not allow a determination of the excitation gap and the difference between $\nu = \frac{1}{2}$ and $\frac{5}{2}$ was not explored in any detail. A first hint at possible adiabatic continuity (AC) between the MR state and the ground state (GS) of a two-body model interaction was mentioned briefly in a subsequent paper by Wen [5], but limited to a single system size $N_{\rm el} = 10$.

Shortly after its discovery, the $\nu = \frac{5}{2}$ state was studied in a tilted magnetic field [6]. Examining the temperature dependence of the longitudinal resistivity $\rho_{xx} \approx \exp-(\Delta/2k_BT)$, the activation gap $\Delta$ was found to decrease with increasing tilt angle and the Hall plateau disappeared beyond some critical tilt angle. These results suggested that the quantized state is at most partially spin polarized until at some critical tilt angle the increasing Zeeman energy produces a phase transition to a gapless polarized state [7].

This scenario was challenged by one of us [8]: exact diagonalization results for small systems on a sphere for spin-unpolarized and fully polarized states at $\nu = \frac{5}{2}$ have shown that the GS is spin polarized even for vanishing Zeeman energy. Furthermore, the GS for $N_{\rm el} = 8$ electrons was found to have substantial overlap with the MR state although that state is the exact ground state of an unphysical short-range three-body interaction Hamiltonian. Subsequent theoretical [9,10] and experimental [11] studies yielded results consistent with these ideas.

These findings led Das Sarma *et al.* [12] to propose the use of the $\nu = \frac{5}{2}$ FQH state for the realization of non-Abelian topological qubits which, they argued, would permit fault tolerant and robust quantum computation. Their proposal prompted a great surge of activity [3] to further elucidate the nature of the $\frac{5}{2}$ FQHE both theoretically [13–16] and experimentally [17–19]. However, whether the FQHE at $\nu = \frac{5}{2}$ observed in experiments has the properties of the non-Abelian MR state remains an open problem, especially since the relevance of the "Pfaffian" state at $\nu = \frac{5}{2}$ has been questioned by [20]: in their exact diagonalization studies of quasiholes (QHS), they only observed QHS with charge $e/2$, while the QHS in the MR state are predicted to have charge $e/4$ [21].

In this Letter we provide theoretical evidence, using state of the art exact diagonalization, that the MR WF and the spin-polarized $\nu = \frac{5}{2}$ FQH state belong to the same universality class [22]. Following the pioneering work by Haldane and Rezayi [23] who established that the GS at $\nu = \frac{1}{3}$ is in the universality class of the $\frac{1}{3}$-Laughlin state, we adiabatically change the electron interaction by interpolating between the three-body interaction $V_{3b}$, for which the Pfaffian WF is the unique GS, and the Coulomb interaction $V_C$ and follow the evolution of GS and energy spectrum by exact numerical diagonalization.

For all even system sizes examined ($N_{\rm el} \leq 18$) we observe AC of the GS and no indication of a decrease of the

 

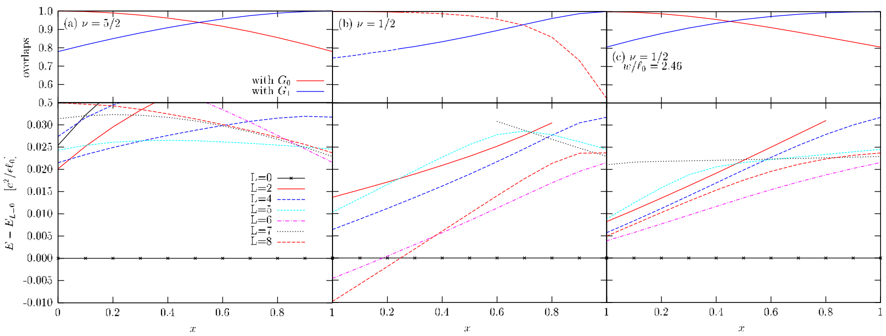


FIG. 1 (color online). $N_{\mathrm{el}} = 16$, $N_\phi = 29$: low-lying energy spectra (the lowest $L = 0$ state is the reference) and overlaps $\langle G_1|G_x\rangle$ and $\langle G_0|G_x\rangle$ as function of the interaction parameter $x$. (a) $\nu = \frac{5}{2}$, (b) $\nu = \frac{1}{2}$, (c) $\nu = \frac{1}{2}$ for finite width $w/\ell_0 = 2.46$. In (b) overlap curves are shown as dashed lines for $x$ values for which the state $|G_x\rangle$ is not the GS.

gap for interactions interpolating between $V_C$ and $V_{3b}$, thus implying AC between the spin-polarized $\frac{5}{2}$ state and the MR state in the thermodynamic limit.

A related study is mentioned in a recent paper by Möller and Simon [16]. They report that in systems of 12, 14, and 16 electrons they see no gap closing when interpolating the interaction between $V_{3b}$ and a particular type of two-body interaction near the Coulomb interaction, but supposedly "in the weak pairing phase" [16]. Contrary to our present Letter, no details are given and the difference between $\nu = \frac{1}{2}$ and $\frac{5}{2}$ is not discussed.

In addition, we systematically vary the two-body interaction, by using the Haldane pseudopotentials determining the pairwise interaction among the electrons, and construct a phase diagram which elucidates the difference between $\nu = \frac{1}{2}$ and $\nu = \frac{5}{2}$ and allows a discussion of the influence of experimental parameters and Landau level mixing on the nature of the state. In this phase diagram the region that corresponds to the gapped phase coincides with the region where the exact numerical GS has a large overlap with the MR WF, further corroborating the AC between the realistic GS and the MR state.

The possibility of MR FQHE at $\nu = \frac{1}{2}$ has first been discussed in [4], but the consensus [24,25] seems to be that the $\nu = \frac{1}{2}$ state in a single-layer 2D system is unlikely to be the MR state. In this Letter, we find that under specific conditions (e.g., the thickness of the layer should be a few magnetic lengths wide) the $\nu = \frac{1}{2}$ state can become adiabatically connected to the MR WF for small systems. Yet for increasing system size, the gap decreases in a way that it is doubtful that the Moore-Read state can occur in single-layer systems at $\nu = \frac{1}{2}$.

In the following we study the low-lying energy spectra of fully spin-polarized systems with up to 18 electrons in the half-filled first ($\nu = \frac{1}{2}$) and second ($\nu = \frac{5}{2}$) LL. If the ground state (GS) has angular momentum $L = 0$ (and is therefore rotation invariant) we consider as "FQH gap" the energy difference between the GS and the first excited state, although the real gap corresponding to an exciton where quasiparticle and quasihole are infinitely separated will be somewhat larger [26].

We use Haldane's spherical geometry [23], in which for a half filled LL the particle number $N_{\mathrm{el}}$ and the number of flux quanta $N_\phi$ are related by $N_\phi = 2N_{\mathrm{el}} - S$. Here, the "shift" $S$ is a topological quantum number [27] and depends on the particular FQH state: for the MR state $S = 3$. We consider particle interactions of the form

$$V = (1 - x)V_{2b} + xV_{3b}, \tag{1}$$

with $0 \le x \le 1$, interpolating between a generic 2-body potential $V_{2b}$ and the 3-body interaction $V_{3b}$ for which the MR WF is an exact GS [5]:

$$V_{3b} = \frac{A}{N_{\mathrm{el}}^5} \sum_{i<j<k}^{N_{\mathrm{el}}} S_{ijk}\{\Delta_j \delta(i - j)\Delta_k^2 \delta(i - k)\}, \tag{2}$$

where $\delta(i - j)$ is the $\delta$ function in the separation of particles $i$ and $j$, $S_{ijk}$ denotes symmetrization over the permutations within the triplet $(ijk)$. We choose the constant $A$ such that $V_{3b}$ generates the approximately same gap as the Coulomb interaction in the second LL. Note that $V_{3b}$ leads to a GS energy which is extensive for any $\nu > \frac{1}{2}$. By projection of $V_{3b}$ on a LL the singularities of the $\delta$ functions are regularized. The 2-body interaction $V_{2b}$ can be written as

$$V_{2b} = \sum_{m=0}^{N_\Phi} v_m \sum_{1 \le i<j \le N_{el}} P_m(ij), \tag{3}$$

where $P_m(ij)$ is the projector on the states in which particles $i$ and $j$ have relative angular momentum $m\hbar$ and the pseudopotential $v_m$ [23] is the corresponding LL-dependent interaction energy.

For $V_{2b}$ we first take the Coulomb interaction $V_C$ of point particles and study the evolution of the ground state $|G_x\rangle$ and lowest energy excited states as function of the parameter $x$. In Fig. 1 we show the energies (lower panel) and overlaps $\langle G_0|G_x\rangle$ and $\langle G_1|G_x\rangle$ (upper panel), where $|G_1\rangle$ is the MR WF (unique GS for $V_{3b}$) and $|G_0\rangle$ is the Coulomb GS. Energies are measured in the usual units $e^2/\epsilon\ell_0$, where $\ell_0 = \sqrt{\hbar c/eB}$ is the magnetic length.

Figure 1(a) shows the results in the second LL: Varying $x$ from 0 to 1 the GS has always angular momentum $L = 0$ and the lowest excitation energy has a weak maximum near $x = \frac{1}{2}$ with values 0.0257, 0.0300, 0.0248, 0.0263, 0.0264, 0.0240 for $N_{\mathrm{el}} = 8$, 10, 12, 14, 16, and 18, respectively. Furthermore, as $x$ is lowered, the overlap with the MR WF slowly decreases reaching a fairly high value at the Coulomb point ($\approx$0.78 for $N_{\mathrm{el}} = 16$ electrons). At the Coulomb point ($x = 0$) and at the Moore-Read point

($x = 1$), we have evidence of a finite gap in the thermodynamic limit (i.e., with infinite quasiparticle-quasihole separation). The gap in the thermodynamic limit has been calculated in [8,26] for Coulomb interaction, while its value for the MR state, with $A = 0.0005$, is about 0.024. Since our calculated gaps for $0 \le x \le 1$ and for all system sizes are never smaller than at the two end points $x = 1$ and $x = 0$, we may expect AC at $\nu = \frac{5}{2}$ between Coulomb GS and MR state in the thermodynamic limit.

The structure of the excitation spectrum (particularly its $L$ dependence) depends on $x$. This must be expected: the interaction between quasiparticles and quasiholes will cause some reordering of excited states. A similar observation was made studying the $\nu = \frac{1}{3}$ state [23].

In the lowest LL at $\nu = \frac{1}{2}$ [see Fig. 1(b)], the situation is different: As $x$ is reduced, at $x = x_c$ we observe a phase transition to compressible GS and for $0 \le x \le x_c$ the GS has angular momentum $L > 0$, and thus vanishing overlap with the Paffian and no AC to the MR state. The situation changes when the finite width of the wave function in the direction perpendicular to the 2D electron system are taken into account. Our results of Fig. 1(c) reveal that a small gap opens down to $x = 0$ and the overlaps are comparable in size or even larger than in the second LL. Thus the finite width induces adiabatic connection between MR WF and Coulomb GS in the lowest LL. In analogy, we also look at the effect of a finite width in the second LL: In agreement with [14], we obtain a decrease of the Coulomb gap, together with an increase of the overlap between MR WF and Coulomb GS. We note LL mixing effects can be accounted for by an effective width in the range $1 \lesssim w/\ell_0 \lesssim 6$, depending on the cyclotron energy (and electron density) [10,28].

To study in detail the finite width effect and the difference between first and second LL we vary the 2-body interaction $V_{2b}$ by changing the pseudopotentials $v_1$ and $v_3$ [Eq. (3)] and keeping all other $v_m$ at their Coulomb values (in a given LL). The values of $v_i$ encode the dependence of the interaction on sample characteristics, like the width of the 2D layer and the electron density.

In Fig. 2(a) we plot the gap as function of $v_1/v_1^{\text{coul}}$ and $v_3/v_3^{\text{coul}}$ for 16 particles in the second LL, where $v_i^{\text{coul}}$ are the Coulomb values of the pseudopotentials. In Fig. 2(b) we do the same for the overlap of the GS with the MR WF. In both cases we find hills with ridges whose positions are close to a straight line given by an approximately fixed $v_3/v_1$ ratio: the gap ridge increases approximately linearly along this line, the overlap ridge rises quickly, reaching values well above 0.9 even for the largest systems ($N_{\text{el}} \le 16$). Remarkably, these two hills are congruent in position for a given system size, while their extent and shape show only little system size dependence. The two hills of gap and overlap thus belong together and are a manifestation of the "MR phase"; below the hills, for smaller $v_3$, we find a compressible phase.

We also note that, if we plot the gaps and overlaps as functions of $y_1 = v_1/v_5$ and $y_3 = v_3/v_5$, the resulting plots for $\nu = \frac{1}{2}$ and $\nu = \frac{5}{2}$ are quite similar, the differences being of the same magnitude as those due to finite size effects. This results from the fact that the higher order $v_m$'s change only little when going from the lowest to the second LL. In Fig. 2(c) we summarize our results for both LL in the $(y_1, y_3)$ plane: the gap contour plot shows the incompressible region, in addition the black line marks the top of the overlap ridge; the shaded (blue) area is the compressible region.

Now looking at the finite width $y_3(y_1)$ trajectories we can view the above results in a new light: for $\nu = \frac{5}{2}$ (thick line marked with 4 dots) the Coulomb point is on the "safe side" of the MR gap ridge, with a consistent gap and a high overlap with the MR WF; increasing the thickness of the system the overlap grows somewhat, as the finite width trajectory approaches the crest of the ridge, while the gap decreases. For $\nu = \frac{1}{2}$ (thick line marked with 5 dots) the

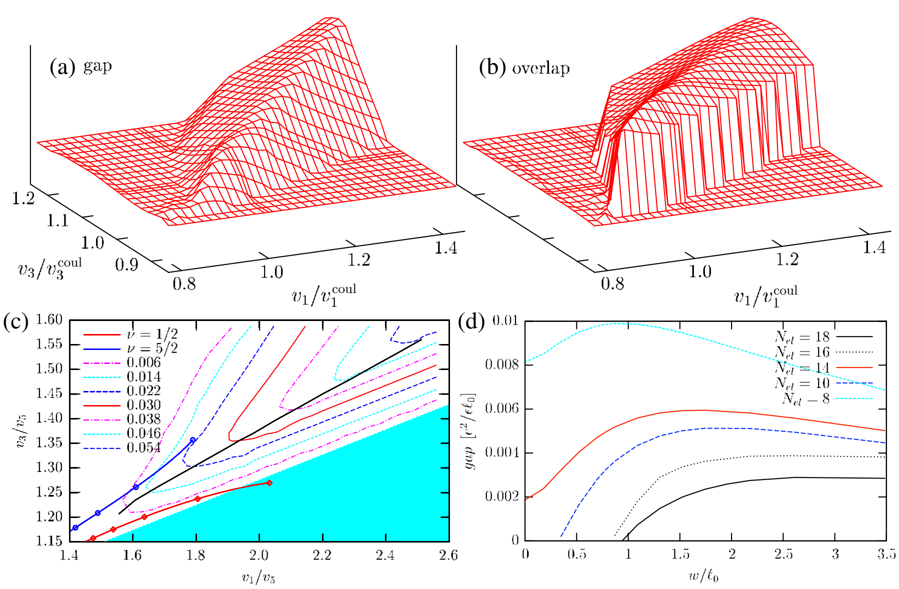


FIG. 2 (color online). Surface plots for (a) gap and (b) overlap of the GS with the MR WF for the 2-body interaction of Eq. (3), varying the pseudopotentials $v_1$ and $v_3$ in the second LL. (c) contour plot of the gap as a function of $v_1/v_5$ and $v_3/v_5$. The black line marks the maxima of the overlap. The thick lines with dots depict the finite width trajectory, the upper (lower) refers to $\nu = \frac{5}{2}(\frac{1}{2})$, the dots denoting width $w/\ell_0 = 0, 1, 2, 3, 4$ (from right to left). The compressible region is shaded blue (all data for $N_{\text{el}} = 16$, $N_\phi = 29$). (d) energy gap at $\nu = \frac{1}{2}$ as function of $w/\ell_0$ for $N_{\text{el}} = 8, 10, 14, 16, 18$.

situation is very different: the Coulomb point is on the other side of the MR ridge, near the line of the phase transition, for some system sizes in the gapped region, for others already in the compressible phase. We thus conclude that the MR phase is so close to the compressible domain that a definite prediction of its existence in the thermodynamic limit is not possible and only experiment can answer.

Indeed, the gaps calculated for $\nu = \frac{1}{2}$ for finite width are small [Fig. 2(d)] and show a marked, although nonmonotonic, decrease with increasing system size ($N_{el} \leq 18$) while the layer width at which the gap opens increases with system size. It is unlikely that a gap survives in the thermodynamic limit for any layer or quantum well width supporting a single-layer system [29]. This proves the importance of careful studies of the system size dependence for valid conclusions about the existence of FQH states.

As a test of our methods in discriminating the MR phase from Abelian FQH phases we studied the system with $N_{el} = 12$ and $N_\phi = 2N_{el} - 3 = 21$, which is "aliased" with the hierarchical $\frac{2}{5}$ state of 10 holes, $N_\phi = \frac{5}{2}N_{holes} - 4 = 21$ [30]. Indeed, our results for the second LL show AC between the Coulomb GS and the MR state. However, in the lowest LL as the interaction is varied from pure three body to Coulomb, the gap increases linearly while the overlap of the MR WF with the GS decreases strongly and its largest overlap is with a high-lying $L = 0$ state ($\Delta E \approx 0.128$): we have entered the Abelian hierarchical phase. The phase transition, as the interaction is varied, from the non-Abelian MR phase to the Abelian hierarchy phase is signaled by a significant and sharp decrease of the overlap between the GS and the MR state. To identify the universality class of a FQH state, AC is thus only a necessary condition, one must also study the overlap between GS and prototype FQH state as well as its system size dependence.

Finally, we address the choice of the shift $S = 3$: three important features characterize states at $S = 3$: (i) the GS at $\frac{5}{2}$ has angular momentum $L = 0$ for all even system sizes $N_{el} \leq 20$ explored by us; (ii) the excitation gap shows a smooth size dependence as expected for a FQH state [8,26]; (iii) low energy states at $S' \neq 3$ can be consistently identified as states with $N_{qp} = \pm 2(S' - 3)$ quasiparticles of charge $\pm e/4$ nucleated in the underlying FQH state with $S = 3$, while the GS has small angular momentum $L = O(N_{el}^0)$ indicating that quasiparticles with charge $\pm e/4$ are well separated.

We have shown that the polarized GS for Coulomb interaction at $\nu = \frac{5}{2}$ is adiabatically connected to the Moore-Read state for all sizes studied. If the gap does not close in the thermodynamic limit—we have not seen any sign that it will—the polarized GS at $\nu = \frac{5}{2}$ has the characteristics of the MR state. The same may happen in the lowest LL at $\nu = \frac{1}{2}$: While finite width and LL mixing effects may help establish a Moore-Read phase, its realization in the thermodynamic limit remains doubtful.

We acknowledge fruitful discussions with N. d'Ambrumenil, J. Fröhlich, and B. I. Halperin and support by the Swiss National Science Foundation and the Institute for Theoretical Physics at ETH, Zurich.

[1] R. Willett *et al.*, Phys. Rev. Lett. **59**, 1776 (1987).
[2] G. Moore and N. Read, Nucl. Phys. **B360**, 362 (1991).
[3] C. Nayak *et al.*, Rev. Mod. Phys. **80**, 1083 (2008).
[4] M. Greiter, X.-G. Wen, and F. Wilczek, Phys. Rev. Lett. **66**, 3205 (1991); Nucl. Phys. **B374**, 567 (1992).
[5] X.-G. Wen, Phys. Rev. Lett. **70**, 355 (1993).
[6] J. P. Eisenstein *et al.*, Phys. Rev. Lett. **61**, 997 (1988); J. P. Eisenstein *et al.*, Surf. Sci. **229**, 31 (1990).
[7] For this reason, it was questionable whether the results of [4] had any relevance for the observed $\nu = \frac{5}{2}$ state.
[8] R. H. Morf, Phys. Rev. Lett. **80**, 1505 (1998).
[9] E. H. Rezayi and F. D. M. Haldane, Phys. Rev. Lett. **84**, 4685 (2000).
[10] R. Morf and N. d'Ambrumenil, Phys. Rev. B **68**, 113309 (2003).
[11] W. Pan *et al.*, Solid State Commun. **119**, 641 (2001).
[12] S. Das Sarma, M. Freedman, and C. Nayak, Phys. Rev. Lett. **94**, 166802 (2005).
[13] Xin Wan, Kun Yang, and E. H. Rezayi, Phys. Rev. Lett. **97**, 256804 (2006); Xin Wan, Zi-Xiang Hu, E. H. Rezayi, and Kun Yang, Phys. Rev. B **77**, 165316 (2008).
[14] M. R. Peterson, Th. Jolicoeur, and S. Das Sarma, Phys. Rev. Lett. **101**, 016807 (2008); Phys. Rev. B **78**, 155308 (2008).
[15] A. E. Feiguin, *et al.*, Phys. Rev. Lett. **100**, 166803 (2008); A. E. Feiguin *et al.*, Phys. Rev. B **79**, 115322 (2009).
[16] G. Moller and S. H. Simon, Phys. Rev. B **77**, 075319 (2008).
[17] I. P. Radu *et al.*, Science **320**, 899 (2008).
[18] M. Dolev *et al.*, Nature (London) **452**, 829 (2008).
[19] C. R. Dean *et al.*, Phys. Rev. Lett. **100**, 146803 (2008); **101**, 186806 (2008).
[20] C. Töke and J. K. Jain, Phys. Rev. Lett. **96**, 246805 (2006); C. Töke, N. Regnault, and J. K. Jain, Phys. Rev. Lett. **98**, 036806 (2007); Solid State Commun. **144**, 504 (2007).
[21] In contrast to [20], our calculations show well separated QHS with charge $e/4$ [R. H. Morf *et al.* (to be published)].
[22] With universality class we mean the class of all states that are adiabatically connected to a prototypal state.
[23] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983); F. D. M. Haldane and E. H. Rezayi, *ibid.* **54**, 237 (1985).
[24] Song He, S. Das Sarma, and X. C. Xie, Phys. Rev. B **47**, 4394 (1993).
[25] Y. W. Suen *et al.*, Phys. Rev. Lett. **72**, 3405 (1994).
[26] R. H. Morf, N. d'Ambrumenil, and S. Das Sarma, Phys. Rev. B **66**, 075408 (2002).
[27] X.-G. Wen and A. Zee, Phys. Rev. Lett. **69**, 953 (1992).
[28] J. Nuebler *et al.*, Phys. Rev. B **81**, 035316 (2010).
[29] B. I. Halperin, P. A. Lee, and N. Read, Phys. Rev. B **47**, 7312 (1993).
[30] N. d'Ambrumenil and R. Morf, Phys. Rev. B **40**, 6108 (1989).