Fast Optimal Frictionless Atom Cooling in Harmonic Traps: Shortcut to Adiabaticity

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A method is proposed to cool down atoms in a harmonic trap without phase-space compression as in a perfectly slow adiabatic expansion, i.e., keeping the same populations of instantaneous levels in the initial and final traps, but in a much shorter time. This may require that the harmonic trap become transiently an expulsive parabolic potential. The cooling times achieved are shorter than those obtained using optimal-control bang-bang methods and real frequencies.

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An ''adiabatic'' process in quantum mechanics is a slow process where the system follows at all times the instantaneous eigenvalues and eigenstates of the time-dependent Hamiltonian. As the populations do not change, there is no heating or friction, but the long times needed may render the operation useless or even impossible to implement. There is currently a surge of interest in adiabatic theory and applications [1–4]. A highly desirable goal is to prepare the same final states and energies of the adiabatic process in a given time t_f, without necessarily following the instantaneous eigenstates along the way. We would also like the procedure to be robust with respect to arbitrary initial states and realizable. If fulfilled, this goal [5] has important implications, since most of the current theoretical and experimental work with cold atoms involves an adiabatic tuning of the system (frequently an expansion or trap weakening) after a cooling phase [1], or as part of the cooling process itself [6]. This adiabatic step has different objectives: the reduction of velocity dispersion and collisional shifts for spectroscopy and atomic clocks [7]; reaching extremely low temperatures inaccessible by standard cooling techniques [6]; or, in experiments with optical lattices, broadening the state before turning on the lattice [8]. These applications would benefit from a shortcut to adiabaticity reducing the times by several orders of magnitude.

The above goal also includes a long-standing question in the fields of optimal-control theory and finite time thermodynamics, namely, to optimize the passage between two thermal states of a system [9–12]. For time-dependent harmonic oscillators, minimal times have been established using ''bang-bang'' real-frequency processes believed up to now to be optimal [11], in which the frequencies are changed suddenly at certain instants but kept constant otherwise. In this Letter we shall describe a robust solution to the stated general goal for atoms trapped in a time-dependent harmonic oscillator which applies both to equilibrium and nonequilibrium states. In particular, we describe cooling processes in which t_f can be smaller than the minimal time of the bang-bang methods considered so far. As well, a general formalism for transitionless dynamics [2] leads for the harmonic oscillator to a nonlocal potential, hindering its realizability, whereas the method proposed here involves a time-dependent, local, realizable potential.

We shall for simplicity describe our method for states representing single atoms of mass m, but the same results are applicable to N-body noninteracting fermions or to a Tonks-Girardeau gas [13], and can be adapted to Bose-Einstein condensates in different dimensions using self-similarity [14]. We consider an effectively one-dimensional time-dependent harmonic oscillator, $H(t) = \hat{p}^2/2m + m\omega^2(t)\hat{q}^2/2$, with an initial angular frequency $\omega(0) > 0$ at time $t = 0$ and final frequency $\omega_f = \omega(t_f) < \omega(0)$ at time t_f. [If the initial and final states are canonical, the corresponding populations and partition functions are the same and the temperature is reduced by a factor $\omega_f/\omega(0)$.] The challenge is to find a trajectory $\omega(t)$ between these two values so that the populations of the oscillator levels $n = 0, 1, 2, \ldots$ at t_f are equal to the ones at $t = 0$. Our main tool to engineer $\omega(t)$ and the state dynamics will be the solution of the corresponding Schrödinger equation based on the invariants of motion [5,15–17] of the form $I(t) = 1/2[(1/b^2)\hat{q}^2 m\omega_0^2 + \frac{1}{m}\hat{\pi}^2]$, where $\hat{\pi} = b\hat{p} - m\dot{b}\hat{q}$ plays the role of a momentum conjugate to $\hat{q}/b$, the dots are derivatives with respect to time, and ω_0 is in principle an arbitrary constant. The scaling, dimensionless function $b = b(t)$ satisfies

$$\ddot{b} + \omega^2(t)b = \omega_0^2/b^3, \tag{1}$$

an Ermakov equation where real solutions must be chosen to make I Hermitian [18]. ω_0 is frequently rescaled to unity by a scale transformation of b [5]. Another convenient choice is $\omega_0 = \omega(0)$ as we shall see. $I(t)$ has the structure of a harmonic oscillator Hamiltonian as well (as long as

$\omega_0^2 > 0$), with time-dependent eigenvectors $|n(t)\rangle$ and time-independent eigenvalues $(n + 1/2)\hbar\omega_0$. The general solution of the Schrödinger equation is a superposition of orthonormal "expanding modes" $\psi(t, x) = \sum_n c_n e^{i\alpha_n(t)}\langle x|n(t)\rangle$, where $\alpha_n(t) = -(n + 1/2)\omega_0 \int_0^t dt'/b^2$, and the c_n are time-independent amplitudes. For a single mode and $\omega_0^2 > 0$,

$$\Psi_n(t, x) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} \frac{e^{-i(n+1/2)\int_0^t dt'(\omega_0/b^2)}}{(2^n n! b)^{1/2}} \times e^{i(m/2\hbar)(\dot{b}/b + i\omega_0/b^2)x^2} H_n\left[\left(\frac{m\omega_0}{\hbar}\right)^{1/2} \frac{x}{b}\right], \quad (2)$$

with time-dependent average energy

$$\langle H(t)\rangle_n = \frac{(2n + 1)\hbar}{4\omega_0}\left(\dot{b}^2 + \omega^2(t)b^2 + \frac{\omega_0^2}{b^2}\right). \quad (3)$$

The average position is zero and the standard deviation $\sigma = (\int dx x^2 |\Psi_n|^2)^{1/2}$ is proportional to b, $\sigma = b(n + 1/2)^{1/2}/(m\omega_0/\hbar)^{1/2}$, which clarifies the physical meaning of the scaling factor.

A much studied case corresponds to the frequency scaling $\omega(t) = \omega(0)/b^2$ with $b = (At^2 + 2Bt + C)^{1/2}$ [15,16]. Substituting this in Eq. (1) gives $\omega_0^2 = \omega(0)^2 + AC - B^2$. For a hard wall trap, the square-root-in-time scaling factor b ($A = 0$) has been shown to provide fast and efficient cooling [19,20]. However, for harmonic traps, much more commonly realized in ultracold experiments, such time dependence leads to negative values of ω_0^2 even for modest cooling objectives, which makes Eq. (2) invalid. Moreover, linear combinations of a continuum of non-square-integrable expanding modes would be needed to describe the evolution of any single eigenstate of the initial trap. Numerical results using other (adiabatic basis) methods [21] show that, even though the square-root-in-time scaling is singularly efficient for adiabatic following, as discussed below, the cooling performance fails for very short expansion times t_f. An alternative, successful strategy put forward here, inspired in inverse scattering techniques for complex potential optimization [22–24], is to leave $\omega(t)$ undetermined at first and impose boundary conditions (BC) on b and its derivatives at $t = 0$ and t_f, to assure that any eigenstate of $H(0)$ evolves as a single expanding mode and that this expanding mode becomes, up to a position-independent phase factor, equal to the corresponding eigenstate of the Hamiltonian $H(t_f)$ of the final trap. In this way the populations in the instantaneous basis will be equal at initial and final times. Once $b(t)$ and its derivatives are fixed at the boundaries, $b(t)$ may be chosen as a real function satisfying the BC, for example, a polynomial or some other convenient functional form with enough free parameters. With $b(t)$ determined, $\omega(t)$ is given by Eq. (1).

Let us first discuss the BC at $t = 0$. By choosing $b(0) = 1$, $\dot{b}(0) = 0$, $H(0)$ and $I(0)$ commute and have common eigenfunctions. We set $\omega_0 = \omega(0)$ from now on so that $\ddot{b}(0) = 0$ must hold as well. These BC imply that any initial eigenstate of $H(0)$, $u_n(0)$ will evolve according to the expanding mode (2) for all later times. In general $H(t)$ and $I(t)$ will not commute for $t > 0$, so that the expanding mode $\Psi_n(t)$ may have more than one component in the "adiabatic basis" of instantaneous eigenstates of $H(t)$, $\{u_n(t)\}$, $n = 0, 1, 2, \ldots$, where $u_n(t) = (\frac{m\omega(t)}{\pi\hbar})^{1/4} \frac{1}{(2^n n!)^{1/2}} \times \exp[-\frac{m}{2\hbar}\omega(t)x^2] H_n[\sqrt{\frac{m\omega(t)}{\hbar}}x]$. At time t_f we want $\Psi_n(t_f)$ to be proportional, up to the global phase factor $e^{i\alpha_n(t_f)}$, to the corresponding eigenstate of the final trap $u_n(t_f)$. Thus we impose $b(t_f) = \gamma = (\omega_0/\omega_f)^{1/2}$, $\dot{b}(t_f) = 0$, $\ddot{b}(t_f) = 0$. From Eq. (3), one finds $\langle H(t_f)\rangle_n$ in terms of $b_f = b(t_f)$ and $\dot{b}_f = db(t)/dt|_{t=t_f}$. Since b_f and $\dot{b}_f$ can be set independently, we can minimize the terms depending on them separately, and the global minimum is found to be precisely at the adiabatic energy $(n + 1/2)\hbar\omega_f$, which corresponds to our BC. Any other choice would necessarily produce "frictional heating."

Substituting the polynomial ansatz $b(t) = \sum_{j=0}^5 a_j t^j$ into the six BC gives six equations that can be solved to provide the coefficients, $b(t) = 6(\gamma - 1)s^5 - 15(\gamma - 1)s^4 + 10(\gamma - 1)s^3 + 1$, where $s = t/t_f$; see Fig. 1. The universality of the solution indicates that there is no fundamental limitation on t_f as long as the potential is truly harmonic, more on this later. At initial and final times 0 and t_f, $\omega(t) = \omega_0/b^2(t)$, but this relation does not hold in general for an arbitrary intermediate time.

The above-mentioned six conditions leave time-dependent phases $e^{i\alpha_n(t)}$ of no relevance regarding the population of the nth level. In particular, stationary density operators with respect to $H(0)$ [e.g., a canonical state, or a pure state $|u_n(0)\rangle\langle u_n(0)|$] are mapped onto the corresponding stationary states of $H(t_f)$ with the phases canceled. In other cases the phases remain, but the populations are preserved. Note that $e^{i\alpha_n(t)}$, see Eq. (2), is the phase factor that the initial state $u_n(0)$ would acquire in a virtual adiabatic process with adiabatic (instantaneous) energy $(n + 1/2)\hbar\omega_0/b^2$. Phase control may also be imposed by adding integral conditions, such as $\tau(t_f) = \int_0^{t_f} dt \frac{1}{b^2(t)} = \frac{\omega_f}{\omega_0} t'$,

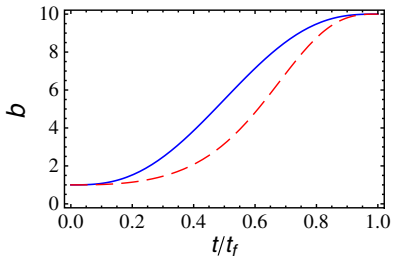

FIG. 1 (color online). Examples of ansatz for b. A simple polynomial ansatz (solid line), and an exponential of a polynomial [dashed line, $\exp(\sum_{j=0}^5 d_j t^j)$]. $\omega(0) = 250 \times 2\pi$ Hz, $\omega(t_f) = 2.5 \times 2\pi$ Hz, $\gamma = 10$.

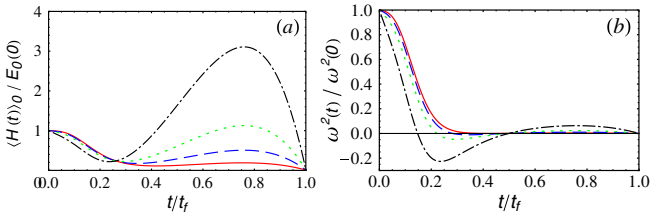

FIG. 2 (color online). (a) The average energies of the ground state expanding mode for different final times t_f: $t_f = 25$ ms (solid line), $t_f = 15$ ms (dashed line), $t_f = 10$ ms (dotted line), and $t_f = 6$ ms (dash-dotted line). Other parameters as in Fig. 1 (polynomial b). (b) The corresponding squared frequency $\omega^2(t)$.

where t' is some desired time. This requires a more complicated ansatz for b, such as a polynomial of higher degree.

Numerical examples of frequencies $\omega(t)$ and energies $\langle H(t)\rangle$ of several expansions that provide a shortcut to adiabaticity are given in Figs. 2–4, using the b shown in Fig. 1 for $\omega_0 = 250 \times 2\pi$ Hz and $\omega_f = 2.5 \times 2\pi$ Hz ($\gamma = 10$). These values can be found in actual experiments [25]. We could formally study subhertz frequencies ω_f, but they would render the trap very sensitive to low-frequency acoustic noise [26]. Compare first the finite times considered in these examples (from 2 to 25 ms) with the times $t_f^{(\mathrm{ad})}$ necessary for actual adiabatic following. The adiabaticity condition for the harmonic oscillator is $|\sqrt{2}\dot{\omega}/(8\omega^2)| \ll 1$. For a linear ramp, $\omega(t) = \omega_0 + (\omega_f - \omega_0)t/t_f$, this implies a very long time, $t_f^{(\mathrm{ad})} \gg 1.1$ s. In practice one would need 6 s to achieve a 1% relative error in the final energy of the ground state with the linear ramp. A much more efficient (still adiabatic) strategy is to distribute $\dot{\omega}/\omega^2$ uniformly along the trajectory, i.e., $\dot{\omega}/\omega^2 = c$, c being constant. Solving this differential equation and imposing $\omega_f = \omega(t_f)$ we get $\omega(t) = \omega_0/[1 - (\omega_f - \omega_0)t/(t_f\omega_f)]$. This corresponds to the case $A = 0$, $2B = -(\omega_f - \omega_0)/(t_f\omega_f)$, $C = 1$ (i.e., a square-root-in-time scaling factor), and implies $t_f^{(\mathrm{ad})} \gg 11$ ms. With this optimized adiabatic trajectory a 1% error level for the ground state energy is achieved after 45 ms.

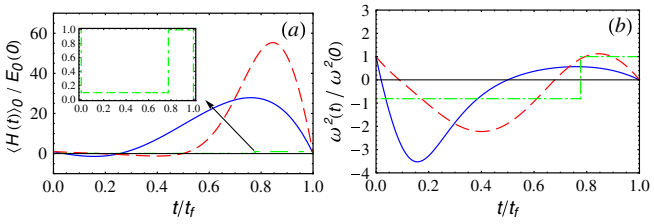

FIG. 3 (color online). Cooling in $t_f = 2$ ms. (a) Average energies of the ground expanding mode for b taken as a polynomial (solid line), as an exponential of a polynomial (dashed line), and for a piecewise constant frequency "bang-bang" process (dot-dashed line) with $\omega_1 = i0.9\omega_0$ and $\omega_2 = \omega_0$. Other parameters as in Fig. 1. (b) The squared frequencies $\omega^2(t)$.

Let us now return to the fast-track trajectories designed with the invariant method. A prominent feature, see Fig. 2(b), is that $\omega^2(t)$ may be negative during some time interval in which the potential becomes an expulsive parabola [27]. In general the (imaginary) frequency of the repulsive region increases for shorter cooling times as shown in Fig. 2(b). A simple estimate for the polynomial ansatz is that the imaginary frequencies occur if $t_f < 1/(2\omega_f)$, the critical time is ≈ 30 ms for the final frequency of the examples. The transient energies below the final one, see, e.g., the solid line in Fig. 3(a) near $t/t_f = 0.15$, are a consequence of the repulsive regime and should not be interpreted as useful cooling in a time shorter than t_f. The kinetic energy would grow without bound if the repeller potential were kept frozen when the energy is minimal. Similarly, if the potential were suddenly changed into its final form, $V(t_f)$, the total energy would be higher than the adiabatic energy.

Figure 3 illustrates that a given cooling objective may be attained in less time than the minimal time required by real-frequency bang-bang trajectories, optimal among real-frequency trajectories [11]. For the three-jump trajectory [11]

$$\omega(t) = \begin{cases} \omega_0 & (t = 0) \\ \omega_1 & (0 < t < \tau_1) \\ \omega_2 & (\tau_1 < t < \tau_1 + \tau_2) \\ \omega_f & (t = t_f = \tau_1 + \tau_2). \end{cases} \tag{4}$$

The fastest process to reach the target state corresponds to $\omega_1 \to 0$ and $\omega_2 \to \infty$ [11] with $t_f^{\min} = \sqrt{1 - \omega_f/\omega_0}/\sqrt{\omega_f\omega_0}$. These results are based on optimal-control theory, initial and final thermal states, and the constraint $\omega_{1,2} > 0$. Clearly, relaxing the positivity condition for the intermediate frequencies makes faster processes with $t_f < t_f^{\min}$ possible, which, moreover, involve only finite frequencies. Since $t_f^{\min}$ has been used to justify a finite time version of the third principle (if $\omega_f \to 0$, $t_f^{\min} \to \infty$ as $\omega_f^{-1/2}$) and maximal cooling rates, the present findings call for a revision of these conclusions. A bang-bang example is shown in Fig. 3 (dot-dashed lines), for $t_f = 2$ ms, much shorter than the time $t_f^{\min} \approx 6$ ms

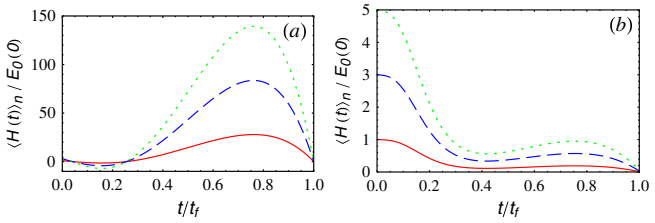

FIG. 4 (color online). Average energies for expanding modes $n = 1$ (solid line), $n = 2$ (dashed line), and $n = 3$ (dotted line). (a) $t_f = 2$ ms and (b) $t_f = 25$ ms. Other parameters as in Fig. 1 (polynomial b).

corresponding to the initial and final frequencies chosen. $\omega_1 = i\omega_I$ is imaginary, and the corresponding $b_1(t)$ is obtained from the Ermakov equation with initial conditions $b_1(0) = 1$ and $\dot{b}_1(0) = 1$. In the second segment we assume ω_2 real and the final conditions $b_2(t_f) = \gamma$ and $\dot{b}_2(t_f) = 0$. The matching conditions $b_1(\tau_1) = b_2(\tau_1)$ and $\dot{b}_1(\tau_1) = \dot{b}_2(\tau_1)$ are then solved for τ_1 and t_f; see Fig. 3 and its caption for details. Of course the discontinuous jumps in this type of trajectory call into question its realizability. Figure 3 also shows two smooth trajectories from the invariant method for $t_f = 2$ ms corresponding to two different ansatz for b, polynomial and exponential of a polynomial. The merits of the polynomial ansatz are its simplicity and the smoothness achieved, but other forms may be found to satisfy further requirements such as, for example, minimizing the maximal frequencies along the trajectory.

The shortcut to adiabaticity using invariants applies to arbitrary initial states, superpositions or mixed. Figure 4 illustrates that the same $\omega(t)$ trajectories used before for the ground state work as well for excited states.

For the experimental realization, we propose the use of a time-dependent far-off resonance optical dipole trap (red detuned) and an antitrap (blue detuned). We have so far considered the 1D case. Formally the three coordinates in an ideal harmonic trap are uncoupled so the expansion processes can be treated independently, but, in our scheme, changing the intensity of a laser beam affects simultaneously the longitudinal and transversal frequencies. Fortunately, the degrees of freedom available, laser intensities and waists [28], are enough to satisfy the desired frequency trajectory in one coordinate, say longitudinal, while keeping the other frequency constant. A second option is to leave the waists constant and add further lasers to compensate for the transversal frequency change. We also have to take into account the anharmonicity and finite depth since they limit the possible excitation of the (initial and final) states. A Gaussian potential $V_0(1 - e^{-2x^2/w^2})$ mimics the harmonic oscillator with frequency $\omega = (2/w)\sqrt{V_0/m}$ holding $\approx V_0/(\hbar\omega)$ bound states. By making t_f smaller the anharmonicity effects become more important. Using time-dependent perturbation theory and the polynomial ansatz for b, we find the condition $w^2 \gg 3\hbar[(n+1)^2 + n^2]/(8mt_f\omega_f^2)$ for a high fidelity of the vibrational state n. Solving numerically the time-dependent Schrödinger equation with the mass of Rb-87, the fidelity for the ground state when $t_f = 2$ ms is 0.91 with $w = 50$ μm, and 0.99 with $w = 150$ μm. It decreases to 0.84 and 0.98 if $t_f = 1$ ms.

As an outlook, similar techniques may be applied to the control of soliton dynamics of Bose-Einstein condensates [27,29], adiabatic computing [3,4], a time-of-flight approach based on fast driven expansions, or in combination with transport of ultracold atoms [30].

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