Closing the Detection Loophole in Bell Experiments Using Qudits

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(Received 8 October 2009; published 11 February 2010)

We show that the detection efficiencies required for closing the detection loophole in Bell tests can be significantly lowered using quantum systems of dimension larger than two. We introduce a series of asymmetric Bell tests for which an efficiency arbitrarily close to $1/N$ can be tolerated using N-dimensional systems, and a symmetric Bell test for which the efficiency can be lowered down to 61.8% using four-dimensional systems. Experimental perspectives for our schemes look promising considering recent progress in atom-photon entanglement and in photon hyperentanglement.

DOI: 10.1103/PhysRevLett.104.060401 PACS numbers: 03.65.Ud, 42.50.Ex

Quantum theory predicts that measurements on separated entangled systems will produce outcome correlations that are not locally causal, as implied by the violation of Bell inequalities [1]. Bell inequality violations have been confirmed in numerous experiments, providing strong indication that nature is nonlocal. However, imperfections in such experiments open various loopholes that could in principle be exploited by a locally causal model to reproduce the experimental data. Given the far-reaching significance of nonlocality, it appears highly desirable to perform a Bell experiment free of any loopholes. While promising proposals have been made recently [2], accompanied by significant experimental progress [3], a loophole-free Bell experiment is still missing.

There are two basic requirements for a loophole-free Bell experiment. First, a spacelike separation between the observers is necessary to ensure that no subluminal signal can propagate between the particles. Failure to satisfy this condition is known as the locality loophole. Second, the particle detection efficiency must be larger than a certain level (usually high); otherwise undetected events can be exploited by a local model to reproduce the quantum statistics [4,5]. Failure to satisfy this condition is known as the detection loophole. All experiments performed so far suffer from at least one of the two above loopholes. Experiments carried out on atoms [3,6] could close the detection loophole, but are unsatisfactory from the locality point of view. Photonic experiments, on the other hand, could close the locality loophole [7], but current photon detection efficiencies are still too low for closing the detection loophole.

Closing the detection loophole is also crucial from the perspective of quantum information processing applications based on quantum nonlocality [8–10]. Indeed, if the detection loophole is not closed, then the data produced in the experiment could as well have been produced by classical means, and thus does not provide any advantage.

For the Clauser-Horne-Shimony-Holt (CHSH) Bell inequality [11], the threshold detection efficiency is 82.8% for a maximally entangled qubit pair and can be lowered down to 66.7% using partially entangled states, as shown by Eberhard [12]. It appears however very difficult to find Bell inequalities that can tolerate lower efficiencies than the CHSH inequality in the bipartite case. For qubits, only marginal improvements have been reported [13–15]. Massar has shown that higher dimensional systems can tolerate a detection efficiency that decrease exponentially with the dimension d, but this result is of limited practical interest since an improvement over the CHSH inequality is obtained only for $d \gtrsim 1600$ [16]. Up to now, no practical bipartite Bell test was known to tolerate a detection efficiency lower than Eberhard's limit of 66.7%.

More recently, the detection loophole has also been studied in an asymmetric configuration [17,18], inspired from atom-photon entanglement. Since atomic measurements are very efficient ($\eta \approx 1$), a much lower efficiency, compared to the symmetric case, can be tolerated for the photon—as low as 43% using qubits and a three-setting Bell inequality [18].

Here, we show that (low dimensional) qudits offer a significant advantage over qubits for closing the detection loophole. For asymmetric Bell tests, we show that an efficiency arbitrarily close to $1/N$ can be tolerated using N-dimensional systems and a family of N-setting Bell inequalities introduced by Collins and Gisin [19]. Our construction is optimal in the sense that a Bell test with N measurement settings cannot tolerate a detection efficiency smaller than $1/N$. In the symmetric scenario, we show that an efficiency as low as 61.8% can be tolerated using four-dimensional states and a four-setting Bell inequality introduced in [14]. To the best of our knowledge, these findings improve significantly over all results in the literature for bipartite Bell tests with a reasonable number of measurement settings and dimensions. Moreover, the prospects of experimental implementations look promising considering recent experimental progress in atom-photon entanglement [20] and in photon hyperentanglement [21–23].

Our constructions also provide simple examples of dimension witnesses [24,25], i.e., Bell-type inequalities which yield a lower bound on the Hilbert space dimension necessary to produce certain quantum correlations.

Preliminaries.—We consider a Bell-type scenario in which two distant parties, Alice and Bob, can choose among N measurement settings with binary outcomes. Alice's settings are denoted A_x with $x \in \{1, \dots, N\}$ and her output bit is denoted $a \in \{+1, -1\}$; similarly for Bob we have B_y with $y \in \{1, \dots, N\}$ and $b \in \{+1, -1\}$. The experiment is characterized by the set of joint probabilities $P(A_x = a, B_y = b)$ to get outcomes a and b when A_x and B_y are measured. All these probabilities are determined by the following subset of probabilities: $P(A_x B_y) \equiv P(A_x = 1, B_y = 1)$, $P(A_x) \equiv P(A_x = 1)$, and $P(B_y) \equiv P(B_y = 1)$, which it is thus sufficient to consider.

The detection efficiencies for Alice and Bob are denoted η_A and η_B. We will study the symmetric configuration, where $\eta \equiv \eta_A = \eta_B$, as in standard photonic experiments, as well as the asymmetric configuration, where $\eta_A = 1$ and $\eta_B < 1$, inspired from atom-photon entanglement.

Asymmetric case.—We now introduce a family of Bell tests that involve entangled states whose local Hilbert space dimension is N and which tolerate a detector efficiency arbitrarily close to $1/N$. Our construction is based on a family of Bell inequalities introduced in [19], which are given by $I_{NN22} \leq 0$, where

$$I_{NN22} = -P(A_1) - \sum_{y=2}^{N} P(B_y) + \sum_{y=1}^{N} P(A_1, B_y) + \sum_{x=2}^{N} P(A_x, B_x) - \sum_{1 \leq y < x \leq N} P(A_x, B_y). \quad (1)$$

Note that we have written these inequalities in a different form than the one used in [19].

Because of the limited efficiency of his detector, Bob does not always obtain a conclusive result. In order to close the detection loophole, one must ensure that the whole set of data, including inconclusive results, violates a Bell inequality. To take into account inconclusive events, we simply choose here that Bob outputs "-1" in case of nondetection. Thus Bob's output is still binary and the above inequalities can be used. The measurement outcome probabilities are however modified according to $P(A_x, B_y) \to \eta_B P(A_x, B_y)$, $P(A_x) \to P(A_x)$, and $P(B_y) \to \eta_B P(B_y)$. Introducing these expressions in (1) and dividing by η_B, we obtain the modified (efficiency-dependent) Bell inequalities $I_{NN22}(\eta_B) \leq 0$, where

$$I_{NN22}(\eta_B) = I_{NN22} - \frac{1 - \eta_B}{\eta_B} P(A_1). \quad (2)$$

A violation of the modified inequality $I_{NN22}(\eta_B) \leq 0$ implies that the original Bell inequality $I_{NN22} \leq 0$ can tolerate a detection efficiency of η_B for Bob's detector.

We now give an explicit entangled state and quantum measurements that violate the inequality $I_{NN22}(\eta_B) \leq 0$ when $\eta_B > \frac{1}{N}$. Note that our construction is optimal in the sense that no violation is possible if $\eta_B \leq \frac{1}{N}$. Indeed, any Bell test with N measurement settings admits a simple local model when $\eta_B \leq \frac{1}{N}$ [26]. The quantum state is defined in $\mathbb{C}^N \otimes \mathbb{C}^N$ and given in the Schmidt form as

$$|\psi_\epsilon\rangle = \sqrt{\frac{1-\epsilon^2}{N-1}} \left(\sum_{k=1}^{N-1} |k\rangle|k\rangle \right) + \epsilon |N\rangle|N\rangle, \quad (3)$$

with $\epsilon \in [0, 1]$. The measurement operators for Alice can be written as $A_x = A_x^+ - A_x^-$, where A_x^+ is the projector on the $+1$ subspace and $A_x^- = I - A_x^+$ on the -1 subspace. We take the projectors A_x^+ to be one-dimensional real-valued projectors, parameterized by the unit vectors $\vec{A}_x \in \mathbb{R}^N$, i.e., $A_x^+ = |a_x\rangle\langle a_x|$, with $|a_x\rangle = \sum_{i=1}^N \vec{A}_{xi} |i\rangle$. The measurement operators for Bob are defined in the same way with unit vectors $\vec{B}_y \in \mathbb{R}^N$.

The measurements of Alice are then defined by

$$\begin{aligned}
\vec{A}_1 &= (0, \dots 0, 0, 0, 1) \\
\vec{A}_2 &= (0, \dots 0, -p_2, \tfrac{p_1}{N-1}, p_0) \\
\vec{A}_3 &= (0, \dots -p_3, \tfrac{p_2}{N-2}, \tfrac{p_1}{N-1}, p_0) \\
&\vdots \\
\vec{A}_{N-1} &= (-p_{N-1}, \dots \tfrac{p_3}{N-3}, \tfrac{p_2}{N-2}, \tfrac{p_1}{N-1}, p_0) \\
\vec{A}_N &= (p_{N-1}, \dots \tfrac{p_3}{N-3}, \tfrac{p_2}{N-2}, \tfrac{p_1}{N-1}, p_0)
\end{aligned} \quad (4)$$

where $p_0^2 = \frac{1}{N}$, $p_1^2 = \frac{N-1}{N}$, and $p_{k+1}^2 = (1 - \frac{1}{(N-k)^2}) p_k^2$ for $k \geq 1$. The measurements of Bob are defined by

$$\begin{aligned}
\vec{B}_1 &= (0, \dots 0, 0, -q_1, q_0) \\
\vec{B}_2 &= (0, \dots 0, -q_2, \tfrac{q_1}{N-1}, q_0) \\
\vec{B}_3 &= (0, \dots -q_3, \tfrac{q_2}{N-2}, \tfrac{q_1}{N-1}, q_0) \\
&\vdots \\
\vec{B}_{N-1} &= (-q_{N-1}, \dots \tfrac{q_3}{N-3}, \tfrac{q_2}{N-2}, \tfrac{q_1}{N-1}, q_0) \\
\vec{B}_N &= (q_{N-1}, \dots \tfrac{q_3}{N-3}, \tfrac{q_2}{N-2}, \tfrac{q_1}{N-1}, q_0)
\end{aligned} \quad (5)$$

where $q_1^2 + q_0^2 = 1$, and $q_{k+1}^2 = (1 - \frac{1}{(N-k)^2}) q_k^2$ for $k \geq 1$. The probabilities entering (2) are then given by

$$\begin{aligned}
&P(A_1) = \epsilon^2 \\
&P(B_y) = \frac{1-\epsilon^2}{N-1}(1 - q_0^2) + \epsilon^2 q_0^2 \quad \text{for } 2 \leq y \leq N \\
&P(A_1, B_y) = \epsilon^2 q_0^2 \quad \text{for } 1 \leq y \leq N \\
&P(A_x, B_x) = \left(\sqrt{\frac{1-\epsilon^2}{N-1}} p_1 q_1 + \epsilon p_0 q_0 \right)^2 \quad \text{for } x \geq 2 \\
&P(A_x, B_y) = \left(\sqrt{\frac{1-\epsilon^2}{N-1}} \frac{p_1 q_1}{1-N} + \epsilon p_0 q_0 \right)^2 \quad \text{for } x > y \geq 1.
\end{aligned} \quad (6)$$

In order to maximize the quantum violation of the inequality (2), we choose the free parameter ϵ defining the quantum state (3) as $\epsilon^2 = \frac{1-q_0^2}{1+[(N-1)^2-1]q_0^2}$, so that the joint probabilities $P(A_x, B_y)$ with $x > y \geq 1$ cancel out. Note that $0 \leq \epsilon \leq 1$ for $0 \leq q_0 \leq 1$. We then find that

$$I_{NN22}(\eta_B) = \epsilon^2\left(-\frac{1}{\eta_B} + q_0^2 N\right). \tag{7}$$

This quantity is positive if $\eta_B > \frac{1}{Nq_0^2}$. Therefore, the inequality $I_{NN22}(\eta_B) \leq 0$ can be violated for any value of $\eta_B > \frac{1}{N}$ by taking q_0 sufficiently close to 1.

In the limit $q_0 \to 1$, we get $\epsilon \to 0$. Thus the state providing the lowest detection efficiency is arbitrarily close to a maximally entangled state of dimension $N-1$. In particular in the qubit case ($N = 2$), it becomes close to a separable state, analogously to the results in [12,17,18].

Note that while the measurement settings (4) and (5) are well adapted for partially entangled states ($\epsilon \to 0$), they do not provide the maximal violation for all values of ϵ. It would be interesting to find a construction that is optimal for any degree of entanglement. For the case $N = 3$, we optimized numerically the unit vectors $\vec{A}_x$ and $\vec{B}_y$ to find the optimal threshold efficiency η_B as a function of the degree of entanglement ϵ (see Fig. 1). Similarly to results in the qubit case [12,17,18], η_B decreases with ϵ. We also investigated the influence of background noise, by replacing the pure state $|\psi_\epsilon\rangle$ in Eq. (3) by a mixed state of the form $\rho = (1-p)|\psi_\epsilon\rangle\langle\psi_\epsilon| + p\mathbb{1}/N^2$, where p is the amount of white noise. For small values of ϵ, η_B becomes quite sensitive to noise, due to the fact that the violation of the inequality $I_{3322}(\eta_B) \leq 0$ becomes small.

We have also investigated [27] the case where the efficiency of Alice's detector is close to (but not exactly) one, which might be relevant from experimental perspectives.

Dimension witnesses.—For the case $N = 3$, we have strong numerical evidence [27] that the inequality $I_{3322}(\eta_B) \leq 0$ cannot be violated by performing measurement on qubits if $\eta_B \leq 0.428$. Therefore the inequality $I_{3322}(\eta_B = 0.428) \leq 0$ is a dimension witness [24]: its violation guarantees that qutrits (at least) have been prepared. We conjecture that the inequalities $I_{NN22}(\eta_B) \leq 0$ with $\eta_B \simeq 1/N$ can only be violated by states of local dimension N, and thus are N-dimensional witnesses.

Symmetric case.—Here we show that the threshold detection efficiency can also be significantly lowered using low dimensional qudits. Specifically, we consider the case $N = 4$ [28] and the Bell inequality $I_{4422}^4 \leq 0$ introduced in Ref. [14], which can be rewritten as

$$\begin{aligned} I_{4422}^4 = {} & I_{\mathrm{CH}}^{(1,2;1,2)} + I_{\mathrm{CH}}^{(3,4;3,4)} - I_{\mathrm{CH}}^{(2,1;4,3)} - I_{\mathrm{CH}}^{(4,3;2,1)} \\ & - P(A_2) - P(A_4) - P(B_2) - P(B_4) \end{aligned} \tag{8}$$

with $I_{\mathrm{CH}}^{(i,j;m,n)} \equiv P(A_i, B_m) + P(A_j, B_m) + P(A_i, B_n) - P(A_j, B_n) - P(A_i) - P(B_m)$. To take into account inconclusive events we choose that Alice and Bob output "-1" in case of nondetection. The probabilities are thus modified according to $P(A_x, B_y) \to \eta^2 P(A_x, B_y)$, $P(A_x) \to \eta P(A_x)$, $P(B_y) \to \eta P(B_y)$. Inserting these values in (8), we obtain, similarly to the asymmetric case, a modified (efficiency dependent) Bell inequality $I_{4422}^4(\eta) \leq 0$.

Next we consider entangled states of the form (3) with $N = 4$ (entangled ququarts), and measurement operators parameterized as previously by unit vectors $\vec{A}_x$ and $\vec{B}_y$ [27]. In the limit $\epsilon \to 0$, we show that the inequality $I_{4422}^4(\eta) \leq 0$ is violated if $\eta > (\sqrt{5}-1)/2 \simeq 0.618$. Using the techniques of [29], we checked that this value is optimal for I_{4422}^4, i.e., that no violation is possible if $\eta \leq 0.618$. For the maximally entangled state ($\epsilon = \frac{1}{2}$), an efficiency of 76.98% can be tolerated. Figure 2 presents the minimal detection efficiency η for intermediate values of ϵ, obtained by performing a numerical optimization over the unit vectors $\vec{A}_x$ and $\vec{B}_y$. These results represent a significant improvement over the best values known so far for small systems, namely, 79.39% for maximally entangled ququarts [13], and 66.67% for weakly entangled qubits [12]. In Fig. 3, we analyze the influence of background noise. For experimentally realistic values of the back-

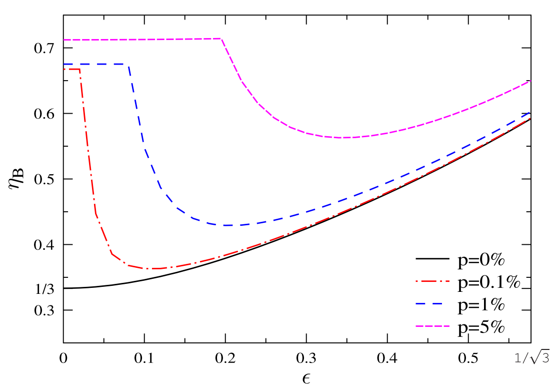

FIG. 1 (color online). Asymmetric Bell test with qutrits. Threshold efficiency η_B as a function of the degree of entanglement ϵ for different values of background noise p. For $p = 0$, the efficiency tends to 33.3% when $\epsilon \to 0$, and is equal to 66.7% at the value $\epsilon = 0$ (i.e., for maximally entangled qubits). For $p > 0$, the curve exhibits a plateau when the efficiency becomes equal to the one given by maximally entangled qubits.

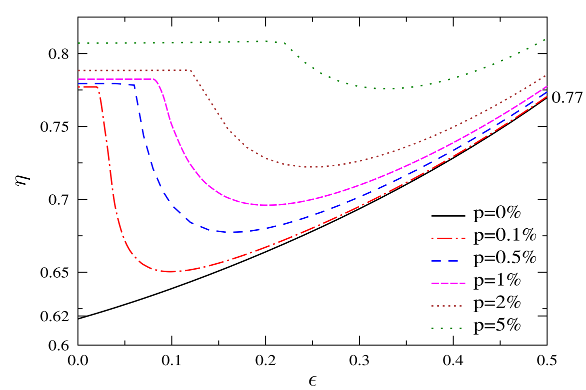

FIG. 2 (color online). Symmetric Bell test with four-dimensional systems. Threshold efficiency η as a function of the degree of entanglement ϵ for different values of background noise p. For maximally entangled states ($\epsilon = \frac{1}{2}$) an efficiency of 76.98% can be tolerated; for very partially entangled states ($\epsilon \to 0$), the efficiency drops down to 61.8%.

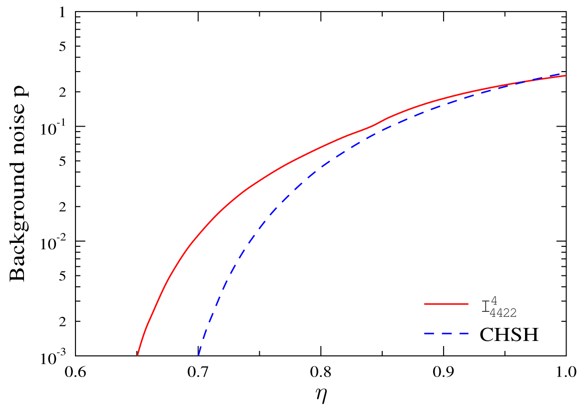

FIG. 3 (color online). Minimal detection efficiency η for a given amount of noise p. The curve for the CHSH inequality is Eberhard's result [12]. For a realistic amount of background noise ($p < 2\%$), our models can tolerate efficiencies 5%–8% lower than the CHSH inequality.

ground noise ($p < 2\%$), our model can tolerate efficiencies about 5%–8% lower compared to the CHSH inequality.

Experimental perspectives.—In the asymmetric case, related to atom-photon entanglement, we showed that an efficiency arbitrarily close to $1/N$ can be tolerated for the photon, using N-dimensional quantum states. Therefore, the (typically) low photon detection efficiency, usually the main experimental limitation, can be compensated by increasing the Hilbert space dimension, which appears feasible. Notably, qutrit entanglement has been recently observed between an atomic ensemble and a photon [20]. Since the experiment [20] used orbital angular momentum as the degree of freedom, it could in principle be performed with higher dimensional systems as well.

For the symmetric case, we showed that efficiencies as low as 61.8% can be tolerated using four-dimensional states, thus providing a significant improvement over the CHSH inequality—the best inequality known so far. This appears promising from the perspective of photon experiments, in which higher dimensional entanglement has been demonstrated using orbital angular momentum [30], time bins [31], multipath entanglement [32], and hyperentanglement [21], i.e., photons entangled in several degrees of freedom [22]; see [33] for a review. Hyperentanglement appears here particularly relevant, since four-dimensional entangled states can be conveniently obtained using photons entangled in both polarization and mode [23]. Partially entangled states of the form (3), as well as the required measurements, can be implemented using standard equipment [27]. Moreover, since the measurements involved in our model have binary outcomes, only one detector is required on each side. Considering a realistic background noise of 1% (i.e., visibilities of 99%), the required overall efficiency, including transmission from the source to Alice (or Bob) and photon detection, is $\sim$69% (see Fig. 3). For a measurement time of 100 ns, the distance between the source and Alice (Bob) should be of the order of 50 meters, to ensure spacelike separation. On such distances transmission losses are negligible. The main limitation is the collection of photons from the source. Still, collection efficiencies higher than 80% have been achieved [34], which imposes a photon detection efficiency greater than 86%. This is a demanding feature, but efficiencies of the order of 90% have already been reported [35]. Thus, overall, the perspective of implementing a loophole-free experiment using photon hyperentanglement seems very promising.

We thank N. Gisin, E. Pomarico, J. Rarity, V. Scarani, and H. Zbinden for discussions, and acknowledge financial support from the Swiss SNSF and NCCR Quantum Photonics, the ERC-AG QORE, the UK EPSRC, and from a János Bolyai Programme of the Hungarian Academy of Sciences.

[1] J. Bell, Physics **1**, 195 (1964).
[2] C. Simon and W. T. M. Irvine, Phys. Rev. Lett. **91**, 110405 (2003); R. Garcia-Patron *et al.*, Phys. Rev. Lett. **93**, 130409 (2004); W. Rosenfeld *et al.*, Adv. Sci. Lett. **2**, 469 (2009); M. Stobinska *et al.*, arXiv:0909.1545.
[3] D. N. Matsukevich *et al.*, Phys. Rev. Lett. **100**, 150404 (2008).
[4] P. M. Pearle, Phys. Rev. D **2**, 1418 (1970).
[5] B. Gisin and N. Gisin, Phys. Lett. A **260**, 323 (1999).
[6] M. A. Rowe *et al.*, Nature (London) **409**, 791 (2001).
[7] A. Aspect, Nature (London) **398**, 189 (1999).
[8] R. Cleve and H. Buhrman, Phys. Rev. A **56**, 1201 (1997).
[9] A. Acin *et al.*, Phys. Rev. Lett. **98**, 230501 (2007).
[10] C.-E. Bardyn *et al.*, Phys. Rev. A **80**, 062327 (2009).
[11] J. F. Clauser *et al.*, Phys. Rev. Lett. **23**, 880 (1969).
[12] P. H. Eberhard, Phys. Rev. A **47**, R747 (1993).
[13] S. Massar *et al.*, Phys. Rev. A **66**, 052112 (2002).
[14] N. Brunner and N. Gisin, Phys. Lett. A **372**, 3162 (2008).
[15] K. F. Pal and T. Vertesi, Phys. Rev. A **79**, 022120 (2009).
[16] S. Massar, Phys. Rev. A **65**, 032121 (2002).
[17] A. Cabello and J.-A. Larsson, Phys. Rev. Lett. **98**, 220402 (2007).
[18] N. Brunner *et al.*, Phys. Rev. Lett. **98**, 220403 (2007).
[19] D. Collins and N. Gisin, J. Phys. A **37**, 1775 (2004).
[20] R. Inoue *et al.*, Phys. Rev. Lett. **103**, 110503 (2009).
[21] P. Kwiat, J. Mod. Opt. **44**, 2173 (1997).
[22] J. T. Barreiro *et al.*, Phys. Rev. Lett. **95**, 260501 (2005).
[23] M. Barbieri *et al.*, Phys. Rev. A **75**, 042317 (2007).
[24] N. Brunner *et al.*, Phys. Rev. Lett. **100**, 210503 (2008).
[25] T. Vertesi and K. F. Pal, Phys. Rev. A **77**, 042106 (2008).
[26] S. Massar and S. Pironio, Phys. Rev. A **68**, 062109 (2003).
[27] See supplementary material at http://link.aps.org/supplemental/10.1103/PhysRevLett.104.060401.
[28] For $N = 3$, there is no improvement for the minimal detection efficiency over CHSH (i.e., $\eta > 2/3$).
[29] M. Navascués *et al.*, New J. Phys. **10**, 073013 (2008).
[30] G. Molina-Terriza *et al.*, Nature Phys. **3**, 305 (2007).
[31] R. T. Thew *et al.*, Phys. Rev. Lett. **93**, 010503 (2004).
[32] A. Rossi *et al.*, Phys. Rev. Lett. **102**, 153902 (2009).
[33] M. Genovese and P. Traina, Adv. Sci. Lett. **1**, 153 (2008).
[34] T. B. Pittman *et al.*, Opt. Commun. **246**, 545 (2005).
[35] S. Takeuchi *et al.*, Appl. Phys. Lett. **74**, 1063 (1999); D. Rosenberg *et al.*, Phys. Rev. A **71**, 061803(R) (2005).