P H Y S I C A L   R E V I E W   L E T T E R S

$\mathcal{PT}$-Symmetric Wave Chaos

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We study a new class of chaotic systems with dynamical localization, where gain or loss mechanisms break the Hermiticity, while allowing for parity-time ($\mathcal{PT}$) symmetry. For a value $\gamma_{\mathcal{PT}}$ of the gain or loss parameter the spectrum undergoes a spontaneous phase transition from real (exact phase) to complex values (broken phase). We develop a one parameter scaling theory for $\gamma_{\mathcal{PT}}$, and show that chaos assists the exact $\mathcal{PT}$ phase. Our results have applications to the design of optical elements with $\mathcal{PT}$ symmetry.

Introduction.—Systems exhibiting parity-time ($\mathcal{PT}$) symmetry have been the subject of rather intense research activity during the last few years. This interest was motivated by various areas of physics, ranging from quantum field theories and mathematical physics [1–4] to solid state physics [5,6] and classical optics [7–13]. A surprising result that was pointed out in some of these investigations was the possibility that $\mathcal{PT}$ symmetric Hamiltonians $\mathcal{H}$ can have real spectrum, despite the fact that they can, in general, be non-Hermitian [3]. The departure from Hermiticity is due to the presence of various gain or loss mechanisms which occur in a balanced manner, so that the net loss or gain of "particles" is zero. Furthermore, as some gain or loss parameter γ that controls the degree of non-Hermiticity of $\mathcal{H}$ changes, a spontaneous $\mathcal{PT}$ symmetry breaking occurs. At this point, $\gamma = \gamma_{\mathcal{PT}}$, the eigenfunctions of $\mathcal{H}$ cease to be eigenfunctions of the $\mathcal{PT}$ operator, despite the fact that $\mathcal{H}$ and the $\mathcal{PT}$ operator commute [3]. This happens because the $\mathcal{PT}$ operator is antilinear, and thus the eigenstates of $\mathcal{H}$ may or may not be eigenstates of $\mathcal{PT}$. As a consequence, in the broken $\mathcal{PT}$ symmetry phase the spectrum becomes partially or completely complex. The other limiting case where both $\mathcal{H}$ and $\mathcal{PT}$ share the same set of eigenvectors, corresponds to the so-called exact $\mathcal{PT}$ symmetric phase in which the spectrum is real.

A promising realization of $\mathcal{PT}$ symmetric systems appears in the frame of optics, where a medium with alternating regions of gain and loss can be synthesized, such that the (complex) refraction index satisfies the condition $n^*(-x) = n(x)$ [6–11]. This kind of synthetic $\mathcal{PT}$ materials exhibits unique characteristics such as "double refraction" and nonreciprocal diffraction patterns, which may allow their use as a new generation of unidirectional optical couplers or left-right sensors of propagating light [9]. Recently, the interest in $\mathcal{PT}$ systems bursted further due to their experimental realization [12,13]. In this respect, one of the emerging questions is how one can enhance the parameter regime for which exact $\mathcal{PT}$ phase

is present, while at the same time provide a general theoretical formalism for the behavior of $\gamma_{\mathcal{PT}}$, in terms of system parameters like imperfections, system size, complexity of the underlying classical (ray) dynamics, etc.

In this Letter, we investigate the behavior of the exact $\mathcal{PT}$ phase in a new setting of systems, namely, a class of Hamiltonians whose classical (ray) dynamics is chaotic while its quantum or wave analogue can show dynamical localization [14,15], a dual phenomenon to Anderson localization appearing in disordered media [16]. As a result of this duality, our study (although performed in the framework of wave chaos systems) is directly relevant to disordered quasi-one-dimensional systems like disordered arrays of optical fibers [17,18]. We have developed a one parameter scaling theory for $\gamma_{\mathcal{PT}}$ and show that it is the only relevant parameter that controls the variation of $\gamma_{\mathcal{PT}}$ with N, the system size of a sample. Specifically,

$$\frac{\partial \tilde{\gamma}_{\mathcal{PT}}}{\partial \log N} = \beta(\tilde{\gamma}_{\mathcal{PT}}); \quad \text{where } \tilde{\gamma}_{\mathcal{PT}} \equiv N \gamma_{\mathcal{PT}} \quad (1)$$

where β is a universal function of $\tilde{\gamma}_{\mathcal{PT}}$ alone.

Furthermore, we have investigated the distribution of the $\mathcal{PT}$ parameter $\mathcal{P}(\gamma_{\mathcal{PT}})$ in the localized and delocalized (chaotic) regimes and found that it reflects the properties of the respective system. Specifically, we have found that $\mathcal{P}(\gamma_{\mathcal{PT}})$ is log normal in the former case, while in the latter it follows a Wigner distribution, reflecting the level repulsion characterizing systems with chaotic or diffusive dynamics. Our results have direct applications not only to coupled optical $\mathcal{PT}$ elements but also to cold atoms moving in a complex $\mathcal{PT}$ potential [19].

Model.—The prototype model of quantum chaos is the celebrated kicked rotor (KR) which exhibits the phenomenon of dynamical localization (DL) [14,15]. Specifically, it has been shown that the quantum suppression of classical diffusion taking place in momentum space is a result of wave interference phenomena similar in nature to the ones responsible for Anderson localization in random media. We study a variation of the kicked rotor (the $\mathcal{PT}$ KR)

[14,20–22], defined by the time-dependent Hamiltonian [23]

$$\mathcal{H} = \frac{p^2}{2} + K_0 V(q) \sum_n \delta(t - nT); \qquad V(q) = \cos(q) + i\gamma q \tag{2}$$

where (q, p) are a pair of canonical variables and $q \in [-\pi, \pi)$ [24]. The kicks have strength K_0 and period T, which we set to unity without loss of generality. In order to avoid any integrable regions in the classical phase space (at $\gamma = 0$) we take $K_0 \gtrsim 5$. We mark that in the framework of geometric optics Eq. (2) describes the propagation of light ray along a chain of optical elements equally placed along the axis of propagation t, in distance T from one another [25,26]. Furthermore, we assume that the elements are purely refractive and ideally thin with a variation only in one transverse direction q. The phase space variable $p = n_0 dq/dt$ is proportional to the slope of the ray, while n_0 is the free-space refractive index.

The wave (quantum) dynamics of this system is described by the one-period evolution operator

$$\mathcal{U} = \exp(-i\hat{p}^2/4\hbar) \exp(-ikV(\hat{q})) \exp(-i\hat{p}^2/4\hbar), \tag{3}$$

where $k = K_0/\hbar$, $\hat{p} = \hbar\hat{l} = -i\hbar d/dq$, and $-N/2 \le l \le N/2$. For $\gamma = 0$, it was found that the eigenfunctions $\psi_l \equiv \langle l|\psi\rangle$ of $\mathcal{U}$, are exponentially localized in momentum space with a localization length $\xi \equiv \lim_{N\to\infty} 1/\sum_l^N |\psi_l|^4 \approx k^2/2$ [14,15]. They are solutions of the eigenvalue problem

$$\mathcal{U}|\psi\rangle = \lambda|\psi\rangle); \qquad \lambda = \exp(-i\epsilon) \tag{4}$$

where the eigenvalues λ are unimodular at $\gamma = 0$, and the phases ϵ are referred to as quasienergies. In the case of $\hbar = 2\pi M/N$ with M, N integers, Eq. (3) defines a dynamical system on a torus. The localization properties of the eigenstates are determined by the scaling parameter $\Lambda = N/\xi$: if $\Lambda \gg 1$ the eigenstates are exponentially localized while if $\Lambda \ll 1$ they are ergodically spread over the momentum space.

PT-breaking scenario.—In the exact *PT* phase (i.e., $\gamma \le \gamma_{PT}$) all eigenvalues are restricted to the unit circle, resulting in a real quasienergy spectrum (see insets of Fig. 1). We find that the mechanism for transition to the broken *PT* phase is a level crossing between the pair of eigenvalues which are closest on the unit circle for $\gamma = 0$. However, rather than splitting into the complex plane symmetrically around the real line as in the case of Hamiltonian systems [6], these pairs split in a logarithmically symmetric manner away from the unit circle (one to the interior, the other to the exterior). Considering the first breaking pair of levels as an isolated two level system [27] we find that the first branching is described by $|\lambda_\pm|^2 \propto \exp(\pm 2\sqrt{\gamma_{PT}^2 - \gamma^2})$. This square root singularity near the bifurcation point is quite universal of an exceptional point and applies both for $\Lambda \gg 1$ and $\Lambda \ll 1$. This is further
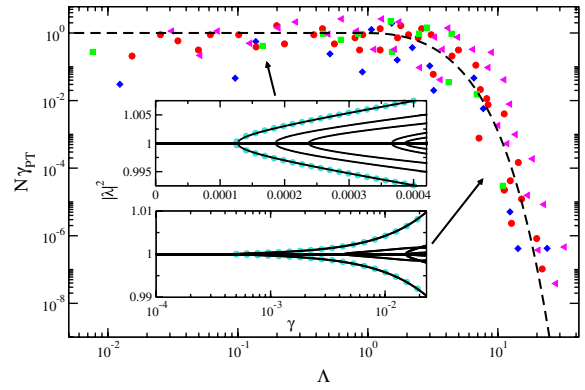

FIG. 1 (color online). Scaling behavior of γ_{PT} for the *PT* KR defined by Eq. (3) for various M, and $K_0 > 5$ with $N = 63$ (pink triangles), 127 (red circles), 225 (green squares), and 511 (blue diamonds). All data show a nice collapse to the theoretical prediction in Eq. (6) (dashed black line). Inset: Parametric evolution of eigenvalue magnitudes for chaos (DL) in the upper (lower) figure with $N = 127$, and $\Lambda = 0.05$ (10). The first branching pair is responsible for the transition to broken *PT* phase, and in both cases follows the predicted functional form (see text) shown with blue circles.

confirmed numerically in Fig. 1 where we present the spontaneous *PT* symmetry breaking scenario for two representative cases associated with localized and delocalized (chaotic) parameter values.

Scaling theory for γ_{PT}.—We consider first the limiting case $\Lambda \gg 1$ where dynamical localization is dominant. To clarify the picture we start from the Hermitian limit $\gamma = 0$. Imagine for the moment that the kicking strength k is zero. Then all states are δ-like functions localized at various momenta $-N/2 \le l \le N/2$. There is an exact degeneracy of multiplicity two between the states localized at $\pm l_0$, i.e., symmetrically around $l = 0$. For $k \ne 0$ this degeneracy is lifted. The eigenstates whose centers of localization are a distance $d = 2l_0 \gg \xi \sim k^2$ apart, form a quasidegenerate pair of symmetric-antisymmetric states [21]. Each has two peaks, near the momenta $\pm l_0$, and decays exponentially $\psi(l) \sim (1/\sqrt{\xi}) \exp(-|\pm l_0 - l|/\xi)$ away from them (double hump states). Thus, the eigenstates in a *P*-symmetric KR are organized into pairs (doublets) ordered by quasienergy difference, $\delta_1 < \delta_2 < \cdots$. The splitting between quasidegenerate levels is $\delta_{N/2-l_0+1} \sim (1/\xi) \exp(-2l_0/\xi)$ while the energy separation between consecutive doublets is much larger, of the order of the mean level spacing of the system, $\Delta \sim 1/N$. Specifically, the smallest energy splitting $\Delta_{\min} = \delta_1 \sim (1/\xi) \times \exp(-N/\xi)$ corresponds to states that were originally located at the extreme points of the momentum "lattice," i.e., $l_0 = \pm N/2$.

As γ is switched on the eigenstates of each pair will initially preserve their *PT*-symmetric structure [6,28]. At $\gamma = \gamma_{PT} \simeq \Delta_{\min} = \delta_1$, the two levels associated with δ_1 will cross, breaking the *PT* symmetry (see Fig. 1 inset). As $\gamma > \gamma_{PT}$ these modes cease to be eigenstates of the

$\mathcal{PT}$ operator. Instead, the weight of each is gradually shifted towards one of the localization centers [6,28]. For larger γ the next doublet (with splitting δ_2) will come into play, creating a second pair of complex eigenvalues for $\gamma \simeq \delta_2$ (see Fig. 1 inset), etc.

Let us now consider the opposite limit of $\Lambda \ll 1$, where the eigenstates are ergodically spread all over the system. In this case, the picture of doublets with exponentially small energy splittings is not valid and $\gamma_{\mathcal{PT}}$ becomes of the order of the minimal level spacing, $\Delta_{\min}$, in the corresponding Hermitian problem. This statement follows from perturbation theory with respect to γ. The unperturbed (i.e., $\gamma = 0$) energy levels are real, and are separated by intervals of order $1/N$, so that $\Delta_{\min} \simeq 1/N$. Finite γ leads to level shifts proportional to γ^2 (the first order correction vanishes due to $\mathcal{PT}$ symmetry) and for $\gamma = \gamma_{\mathcal{PT}} \simeq \Delta_{\min}$ the perturbation theory breaks down, signaling level crossing and the appearance of the first pair of complex eigenvalues. Thus, the energy scale for the $\mathcal{PT}$ threshold in the $N/\xi \ll 1$ limit ($\gamma_{\mathcal{PT}} \simeq 1/N$) widely differs from that for $N/\xi \gg 1$ ($\gamma_{\mathcal{PT}} \simeq (1/\xi)e^{-N/\xi}$).

Combining both cases, we conclude that in the two limits of weak and strong localization we have that

$$\tilde{\gamma}_{\mathcal{PT}} = f(\Lambda) = \begin{cases} \mathcal{O}(1) & \text{for } \Lambda \lesssim 1, \\ \Lambda \exp(-\Lambda) & \text{for } \Lambda > 1. \end{cases} \quad (5)$$

Our numerical results for the $\mathcal{PT}$ KR model, are reported in Fig. 1 and are in excellent agreement with the above theoretical predictions. Moreover, they clearly show that the scaling function $f(x)$ is regular and interpolates smoothly between the two limiting cases. This allows us to conclude that

$$\tilde{\gamma}_{\mathcal{PT}} \equiv N\gamma_{\mathcal{PT}} = f\left(\Lambda \equiv \frac{N}{\xi}\right), \quad (6)$$

which can be rewritten in the form of Eq. (1). This is the main result of the present Letter, as it allows us to postulate the existence of a β function for the $\tilde{\gamma}_{\mathcal{PT}}$ of generic chaotic (or quasi-1D disordered) systems. In the remainder of the Letter, we will focus on the statistical properties of $\gamma_{\mathcal{PT}}$ as a function of the parameter Λ.

Distribution of $\gamma_{\mathcal{PT}}$.—The above discussion pertains only to the behavior of a "typical" system. A full theory, however, must be formulated in statistical terms and deal with probability distributions $\mathcal{P}(\gamma_{\mathcal{PT}})$. To this end we exploit the equivalence between $\gamma_{\mathcal{PT}}$ and $\Delta_{\min}$ which is confirmed numerically in the inset of Fig. 2(a) for Λ values spanning the whole interval from the localized to extended regime. Thus, we instead analyze the distribution $\mathcal{P}(\Delta_{\min})$. For better statistics, an ensemble of $\mathcal{P}$-symmetric KR systems has been created by randomizing the phases of the kinetic part of the evolution operator given by Eq. (3). In all cases, the numerical distribution $\mathcal{P}(\Delta_{\min})$ involved more that 10^4 data points for statistical processing.

We start our analysis from the localized regime $\Lambda \gg 1$. There are several sources of fluctuations in δ_1: fluctuations
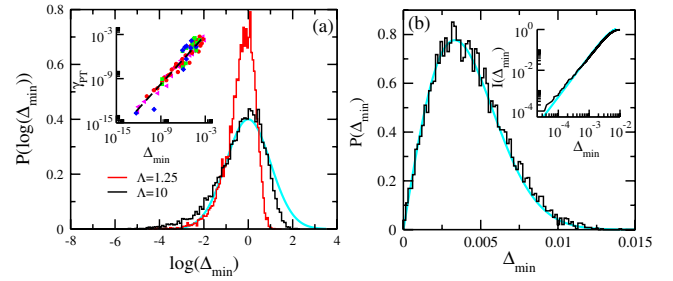

FIG. 2 (color online). (a) Distributions of $\Delta_{\min}$ for localized eigenfunctions displaying a convergence to log-normal behavior. Centers are shifted for ease of comparison. Inset: Linear relation between $\Delta_{\min}$ and γ_{PT} over roughly 12 orders of magnitude. Parameters and symbols are the same as in Fig. 1. (b) Distribution of minimum level spacings in the chaotic regime in which Wignerian behavior (blue line) is observed. Inset: The integrated distribution $I(\Delta_{\min}) = \int_0^{\Delta_{\min}} \mathcal{P}(x)dx$ in a log-log plot. The best fit (blue line) has power two. In both cases, $N = 127$.

in the position and energy of the relevant localized states, as well as what can be termed "fluctuations in the wave functions." By this we mean that a localized wave function exhibits strong, log-normal fluctuations in its "tails," i.e., sufficiently far from its localization center [29]. This latter source of fluctuations appears to be the dominant one and it immediately yields a log-normal distribution for δ_1 (see Fig. 2), since δ_1 is proportional to the overlap integral between a pair of widely separated and strongly localized states [6,29]. We confirm numerically that for increased Λ we do in fact approach such a distribution (see Fig. 2).

In contrast, in the delocalized regime $\Lambda \ll 1$, the structure of the eigenfunctions is random [14,21] and they are ergodically spread over the momentum space. Thus there is no quasidegeneracy for such states; instead, there is a level repulsion due to strong eigenstate overlap. This results in a Wignerian distribution for the minimum energy difference, i.e., $\mathcal{P}(\Delta_{\min}) = (\pi/2)\Delta_{\min} \exp(-\pi\Delta_{\min}^2/4)$, which is dramatically different from the one found in the localized regime [see Fig. 2(a)]. Since $\mathcal{P}(\Delta_{\min}) \sim \mathcal{P}(\gamma_{\mathcal{PT}})$ we conclude that in the case of wave chaos $\mathcal{P}(\gamma_{\mathcal{PT}} \to 0) \to 0$; i.e., there always exists a γ interval for which we will have an exact $\mathcal{PT}$ phase. This observation can be used as a new criterion of wave chaos. Our numerical results for $\Lambda = 0.01$ are shown in Fig. 2(b), and are in excellent agreement with the above theoretical considerations.

Conclusions.—In conclusion, we have studied a new class of quantum chaotic systems with dynamical localization that also possess a $\mathcal{PT}$ symmetry. These systems are described by a non-Hermitian Hamiltonian due to the existence of well-balanced gain or loss mechanisms and show a spontaneous $\mathcal{PT}$ symmetry breaking, i.e., a transition from a real to a complex spectrum, for some value $\gamma_{\mathcal{PT}}$ of the gain or loss parameter γ. We have developed a one-parameter scaling theory for the rescaled critical gain or loss parameter $\tilde{\gamma}_{\mathcal{PT}} = \gamma_{\mathcal{PT}}N$, and conclude that there is a universal β function that depends only on $\tilde{\gamma}_{\mathcal{PT}}$ itself, which controls the variation of $\tilde{\gamma}_{\mathcal{PT}}$ with the system size.

Furthermore, we have analyzed the distribution $\mathcal{P}(\gamma_{\mathcal{PT}})$ in the localized and delocalized regimes, and show that it drastically differs in these two limits. In the former case it is log normal while in the latter it follows a Wigner statistics reflecting the chaoticity of the underlying classical dynamics. Our study opens the way to quantify the spontaneous breaking of the $\mathcal{PT}$ symmetry in terms of universal β function.

The results presented here are based on a simple connection between $\gamma_{\mathcal{PT}}$ and the (minimal) level spacing $\Delta_{\min}$ [see inset of Fig. 2(a)] which is inspired by an isolated two level matrix model (see section on the $\mathcal{PT}$-breaking scenario). Although this is the most generic type of scenario, describing a large number of physical realizations associated with classically chaotic (or quasi-1D disordered) systems, there are many other interesting cases which need to be explored. For example, in the $\mathcal{PT}$-symmetric KR model presented above, we used a discontinuous (at the boundaries) imaginary potential $\text{Im}V(q)$ which gives a lower bound on the behavior of $\gamma_{\mathcal{PT}}$ (worst case scenario). However, preliminary analysis [28] shows that if $\text{Im}V(q)$ is a continuous and analytic function the $\mathcal{PT}$ symmetry breaking scenario can be different. In such a case the matrix elements of $\text{Im}V(q)$ between $\mathcal{P}$ doublets are exponentially small in the regime of dynamical localization. Thus, according to the lowest order perturbation theory, the energy levels of these two states remain nearly parallel (as a function of γ), and hence typically $\gamma_{\text{PT}} \gg \Delta_{\min} = \delta_1$. We also note that one can observe spontaneous antibreaking of $\mathcal{PT}$ symmetry, i.e., for some $\gamma \gg \gamma_{\text{PT}}$, a pair of nonunimodular eigenvalues recombines again into a pair of unimodular eigenvalues, and sometimes (but more rarely as N is increasing) one may even find situations for which all levels simultaneously become unimodular (global recovery of the exact $\mathcal{PT}$ phase) [28]. Understanding of such antibreaking mechanism could be of significant interest for optics applications. Because of lack of space these results will be discuss elsewhere [28].

It will be interesting to extend this line of study to higher dimensions (possibly 3D disordered systems with a metal-to-insulator phase transition). We expect that our study will be of interest not only for the optics community but also for the atomic physics community where complex optical potentials have been recently constructed [19].

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