Anomalous Thermalization of Nonlinear Wave Systems

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We report theoretically and experimentally in an optical system a process of anomalous thermalization of one-dimensional nonlinear Hamiltonian waves. It is characterized by an irreversible evolution of the waves towards a specific equilibrium state of a fundamental different nature than the expected thermodynamic equilibrium state. A kinetic approach of the problem reveals that this phenomenon is due to the existence of a *local invariant* in frequency space. A novel family of equilibrium distributions is discovered, which is found in quantitative agreement with the numerical simulations.

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It has long been established since the pioneering work of Fermi, Pasta, and Ulam (FPU) in the 1950s that a one-dimensional nonlinear chain of particles may not be able to achieve a state of thermal equilibrium, i.e., a state in which the energy is equidistributed among all modes on the average. This fact has important implications for the ergodic hypothesis at the foundation of statistical mechanics. Fundamental mathematical and physical discoveries have led to a better understanding of the FPU problem, although it is by no means completely understood [1].

More recently, the FPU problem and the question of existence of a genuine thermalization process have been revisited in the field of ultracold Bose gases [2], which are known to be described by the nonlinear Schrödinger (NLS) equation, even beyond the zero temperature limit [3]. The NLS equation also provides a universal description of classical nonlinear wave systems, in which FPU recurrences have been interpreted as a (Benjamin-Feir) modulational instability process [4,5]. In spite of the large number of theoretical studies, experimental demonstrations of the FPU recurrences are very rare. They have been observed in deep water waves [6], and, more recently, in magnetic feedback rings [7] and nonlinear optical systems [8]. On the other hand, the asymptotic dynamics leading to the process of thermalization has not been the subject of an accurate experimental demonstration, because irreversible thermalization is predicted in a reversible (lossless) wave system, whereas any realistic system exhibits dissipation.

The process of thermalization of a nonlinear wave system is known to be characterized by an irreversible evolution of the wave towards the thermodynamic Rayleigh-Jeans (RJ) equilibrium distribution [3,9–11]. We show in this work that, in contrast with this commonly accepted picture of thermalization, a one-dimensional wave system may exhibit a process of anomalous thermalization. It is characterized by an irreversible evolution of the waves towards a specific equilibrium state, which is of a fundamental different nature than the conventional RJ equilibrium states. The kinetic wave theory reveals that the anomalous thermalization is due to the existence of a local invariant in frequency space J_ω which originates in degenerate resonances of the system. In contrast to conventional integral invariants that lead to a generalized RJ distribution, here, it is the local nature of the invariant J_ω that makes the new equilibrium states fundamentally different than the usual RJ equilibrium states. The anomalous thermalization is characterized by a process of entropy production: the novel family of equilibrium states is associated to a maximum of the nonequilibrium entropy subject to the additional constraint $J_\omega = \text{const}$. Experiments realized with optical waves provide a signature of the transient process of this anomalous thermalization.

The first model we consider is the vector NLS equation, which is known to be relevant for the description of vector phenomena in optics [12], plasma [13], hydrodynamics [14], or Bose-Einstein condensates [15]

$$i\partial_z A_1 = -\partial_t^2 A_1 + (|A_1|^2 + \kappa|A_2|^2)A_1, \tag{1}$$

$$i\partial_z A_2 = -\eta\partial_t^2 A_2 + (|A_2|^2 + \kappa|A_1|^2)A_2. \tag{2}$$

As usual in optics, the distance z of propagation in the nonlinear medium plays the role of an evolution "time" variable, while t measures the time in a reference frame moving with the waves [12]. We normalized the problem with respect to the nonlinear length $L_0 = 1/(\gamma e_0^2)$ and time $\tau_0 = (\alpha_1 L_0)^{1/2}$, where γ is the nonlinear coefficient, α_j the dispersion coefficients of A_j, and e_0^2 the mean intensity of A_1. With these units, η denotes the ratio between the dispersion coefficients, and κ the ratio between the cross- and self-interaction coefficients. The dispersion relations of $A_{1,2}$ read $k_1(\omega) = \omega^2$, $k_2(\omega) = \eta\omega^2$. Note that Eqs. (1) and (2) conserve the "power" $N_j = \int |A_j|^2 dt$ of each component A_j and the Hamiltonian H [12]. In the particular cases $\eta = \kappa = 1$ (or $\eta = \kappa = -1$), Eqs. (1) and (2) are known to be integrable [12,16]. In this work we shall consider the nonintegrable case.

A physical insight into anomalous thermalization may be obtained by integrating numerically the NLS equations (1) and (2). The initial conditions refer to partially coherent

waves with a Gaussian spectrum and random spectral phases; i.e., $A_{1,2}(z=0,t)$ are of zero mean and obey a stationary statistics. The numerical simulations reveal that, after a complex transient characterized by a spectral broadening, the two waves reach a (statistical) stationary state; i.e., the averaged spectra no longer evolve in the propagation. The simulations show that such stationary states are of a fundamental different nature depending on the value of the dispersion coefficient η in Eqs. (1) and (2). For $\eta \neq 1$ one recovers the standard thermalization process, which is characterized by an irreversible evolution of the fields towards the RJ equilibrium spectrum, whose tails satisfy the property of energy equipartition [see Fig. 1(a)] [10,11]. We observed, however, a breakdown of energy equipartition for $\eta = 1$. This is illustrated in Fig. 1(a), where the energy in frequency space, $\epsilon_j(\omega) = k_j(\omega) n_j(\omega)$, is not equally distributed among the modes, and thus the spectra do not exhibit the expected RJ power-law behavior, $n_j(\omega) \sim k_j(\omega)^{-1} \sim \omega^{-2}$.

We show below that this anomalous thermalization may occur in a large variety of nonlinear systems, whenever they exhibit degenerate resonances. To grasp this anomalous thermalization process, we make use of wave turbulence theory [10], which is based on a natural asymptotic closure of the moments' equations induced by the dispersive properties of the waves. Despite the formal reversibility of Eqs. (1) and (2) the wave turbulence theory derives a set of irreversible kinetic equations that govern the evolutions of the averaged spectra of the fields $n_j(\omega, z)$, with $\langle a_j(\omega_1, z) a_j^*(\omega_2, z)\rangle = n_j(\omega_1, z)\delta(\omega_1 - \omega_2)$, a_j being the Fourier transform of A_j [9,10]:

$$\partial_z n_j(\omega, z) = \int d\omega_1 d\omega_2 d\omega_3 W \mathcal{N}, \tag{3}$$

where $\mathcal{N} = n_j(\omega) n_i(\omega_1) n_i(\omega_2) n_j(\omega_3)[n_j^{-1}(\omega) + n_i^{-1}(\omega_1) - n_i^{-1}(\omega_2) - n_j^{-1}(\omega_3)]$. The resonant conditions of energy and momentum conservation are expressed by the Dirac δ functions in $W = \frac{\kappa^2}{\pi}\delta(\omega + \omega_1 - \omega_2 - \omega_3)\delta[k_j(\omega) + k_i(\omega_1) - k_i(\omega_2) - k_j(\omega_3)]$. Equations (3) conserve the (quasi)particle number of each field A_j, $N_j = \int n_j(\omega) d\omega$ and the kinetic contribution to the energy $E = \sum_i E_i$, $E_i = \int k_i(\omega) n_i(\omega) d\omega$. The irreversible character of Eq. (3) is expressed by a H theorem of entropy growth, $d\mathcal{S}/dz \geq 0$, with $\mathcal{S} = \sum_i \mathcal{S}_i$, $\mathcal{S}_i(z) = \int \log(n_i) d\omega$ being the nonequilibrium entropy. The thermodynamic equilibrium spectra $n_j^{\mathrm{RJ}}(\omega)$ realizing the maximum of $\mathcal{S}[n_j]$, given the constraints of conservation of E and N_j, refer to the well-known RJ distribution

$$n_j^{\mathrm{RJ}}(\omega) = T/[k_j(\omega) - \mu_j], \tag{4}$$

where T denotes the temperature and μ_j the chemical potential of A_j [10,11]. The tails of the RJ spectrum (4) verify the property of energy equipartition, $n_j^{\mathrm{RJ}} \sim \omega^{-2}$, which is in contrast with the numerical results discussed in Fig. 1(a) for $\eta = 1$. However, we shall see below that the kinetic equations (3) still provide a detailed description of the anomalous thermalization process.

Assuming $\eta = 1$, the δ functions in Eq. (3) may be used to compute two integrations. The coupled kinetic equations then reveal the existence of a new invariant, $J(\omega) = n_1(\omega, z) + n_2(\omega, z)$, i.e., $\partial_z J_\omega = 0$. This invariant is "local" in the sense that it holds for each frequency ω individually. The invariant J_ω allows us to derive a closed equation for the evolution of n_1,

$$\partial_z n_1(\omega, z) = \frac{1}{2}\int \frac{\mathcal{G}[J, n_1]}{|\omega - \omega_1|} d\omega_1, \tag{5}$$

where $\mathcal{G}[J, n_1] = \frac{\kappa^2}{\pi}[J_{\omega_1} - n_1(\omega_1)][J_\omega - n_1(\omega)] \times [n_1(\omega_1) - n_1(\omega)] - n_1(\omega_1) n_1(\omega)[J_{\omega_1} - n_1(\omega_1) - J_\omega + n_1(\omega)]$. Note that the singularity in Eq. (5) is apparent only because the function $\mathcal{G}$ also vanishes for $\omega = \omega_1$. This equation conserves $N_1 = \int n_1(\omega, z) d\omega$ and exhibits a H theorem for $\mathcal{S}_{\mathrm{loc}}(z) = \int \log\{n_1(\omega)[J_\omega - n_1(\omega)]\} d\omega$ (note that the conservation of energy E and momentum P is implicitly verified through the invariant J_ω). The spectrum of local equilibrium that realizes the maximum of $\mathcal{S}_{\mathrm{loc}}$ given the constraint of conservation of N_1 is obtained by introducing the Lagrange's multiplier λ,

$$n_1^{\mathrm{loc}}(\omega) = J_\omega/2 - [\sqrt{1 + (\lambda J_\omega/2)^2} - 1]/\lambda, \tag{6}$$

while $n_2^{\mathrm{loc}}(\omega) = J_\omega - n_1^{\mathrm{loc}}(\omega)$. The local equilibrium state (6) is a stationary solution of Eq. (5). The parameter λ is determined by the initial condition through J_ω, $\int(\sqrt{1 + \lambda^2 J_\omega^2/4} - 1) d\omega = \lambda(N_2 - N_1)/2$. If $N_1 = N_2$, one obtains $\lambda = 0$ and the equilibrium spectrum (6) reduces to $n_1^{\mathrm{loc}}(\omega) = n_2^{\mathrm{loc}}(\omega) = J_\omega/2$. In the limit $N_1 \gg N_2$ ($N_1 \ll N_2$), $\lambda \to -\infty$ ($+\infty$), and $n_1^{\mathrm{loc}}(\omega) \to J_\omega$ ($n_2^{\mathrm{loc}}(\omega) \to J_\omega$). Contrary to the RJ distribution (4), the local equilibrium state (6) preserves a memory of the initial condition through the invariant J_ω.

Let us underline the deep difference between the local invariant J_ω and the integral invariants investigated in

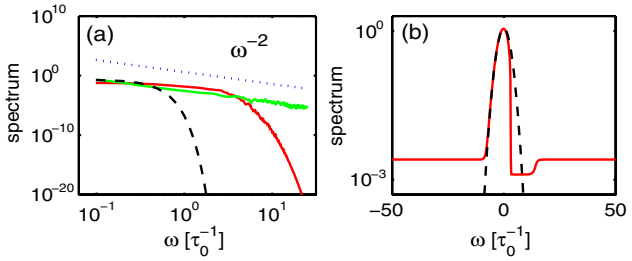

FIG. 1 (color online). (a) Equilibrium spectrum of A_1 obtained by solving numerically the NLS equation (1) and (2) for $\eta = 1$ (red or gray), $\eta = 1.1$ (green or light gray) and corresponding initial condition (dashed dark). The dotted blue line shows the energy equipartition power law $n_j^{\mathrm{RJ}}(\omega) \sim \omega^{-2}$ (b) Local equilibrium spectrum $n^{\mathrm{loc}}(\omega)$ (red or gray) associated to the scalar NLS equation (7), the dashed dark line shows the initial condition, $\alpha = 0.1$.

Refs. [16,17] in line with the problem of integrability. Note in this respect that the possible existence of a set of additional integral invariants, $Q_j = \int \varphi_j(\omega) n_\omega d\omega$, would still lead to a (generalized) RJ distribution, $n^{\mathrm{RJ}}(\omega) = T/[k(\omega) + \sum_j \lambda_j \varphi_j(\omega) - \mu]$, λ_j being the Lagrangian multipliers of Q_j [17]. In contrast, the local invariants J_ω identified here lead to an equilibrium spectrum $n^{\mathrm{loc}}(\omega)$ of a fundamental different nature than $n^{\mathrm{RJ}}(\omega)$.

The numerical integration of the kinetic Eq. (5) reveals an irreversible evolution of $n_1(z, \omega)$ towards the local equilibrium state (6). As illustrated in Fig. 2, this anomalous thermalization has been confirmed by the numerical integration of the NLS equations (1) and (2), in which a quantitative agreement has been obtained with the theory [Eq. (6)], without any adjustable parameter.

The family of local equilibrium states (6) is parametrized by the function J_ω; i.e., for each J_ω we have a different $n_j^{\mathrm{loc}}(\omega)$. Thus a set of distinct equilibrium states can be reached by starting from different initial conditions: in this loose sense the system exhibits a "multistable" behavior. The RJ distribution also belongs to this family; i.e., if $J_\omega = n_1^{\mathrm{RJ}}(\omega) + n_2^{\mathrm{RJ}}(\omega)$ and $\lambda = (\mu_2 - \mu_1)/T$, Eq. (6) gives $n_j^{\mathrm{loc}}(\omega) = n_j^{\mathrm{RJ}}(\omega)$ for $\eta = 1$. This is consistent with the fact that the RJ distribution is associated to the maximum of $\mathcal{S}$ without the constraint $J_\omega = \mathrm{const}$. As a consequence of this "multistability," the system would not recover the initial state by sweeping η back and forth between $\eta = 1$ and $\eta \neq 1$.

The existence of the local invariant J_ω finds its origin in degenerate resonances: the resonant conditions of energy and momentum conservation expressed by W in Eq. (3) exhibit the trivial solution $\omega_{3,4} = \omega_{1,2}$ for $\eta = 1$. Degenerate resonances occur in a great variety of nonlinear wave systems. They occur in the resonant FWI, $i(\partial_z + v_j^{-1}\partial_t)A_j = A_k A_l A_m^* + A_j|A_j|^2 + \kappa A_j \sum_{i\neq j}|A_i|^2$, if $v_1 = v_4$, $v_2 = v_3$. The corresponding local equilibrium spectra read $n_{1,2}^{\mathrm{loc}}(\omega) = J_{1,2}(\omega)/2 - [\sqrt{1 + \lambda^2[J_{1,2}(\omega)]^2/4} - 1]/\lambda$, $n_{3,4}^{\mathrm{loc}}(\omega) = J_{1,2}(\omega) - n_{2,1}^{\mathrm{loc}}(\omega)$, where $J_{1,2}(\omega)$ refer to local invariants.

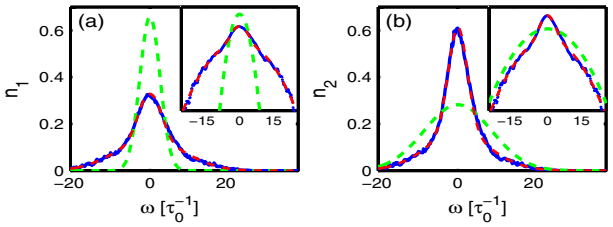

FIG. 2 (color online). Equilibrium spectra obtained by solving Eqs. (1) and (2) numerically (blue or dark gray) and corresponding $n^{\mathrm{loc}}(\omega)$ given by Eq. (6) (dashed red or gray). The dashed green or light gray lines show the initial conditions. The insets show the same plots in log scale. ($\eta = 1$, $N_1/N_2 = 0.7$, an average over $10^5 L_0$ has been taken once equilibrium was reached, $\partial_z \mathcal{S} \simeq 0$.)

We emphasize that anomalous thermalization also takes place with nontrivial degenerate resonances. We illustrate this aspect with the scalar NLS equation, whose integrability is broken by the third-order dispersion

$$i\partial_z A = -\partial_t^2 A - i\alpha\partial_t^3 A + |A|^2 A, \tag{7}$$

where $k(\omega) = \omega^2 - \alpha\omega^3$. The kinetic equation of Eq. (7) reveals the existence of the local invariant $J_\omega = n(z, \omega) - n(z, q - \omega)$, where $q = 2/3\alpha$. It originates in the following degenerate resonance: the frequencies $(\omega, q - \omega)$ resonate with any pair of frequencies $(\omega', q - \omega')$, because $k(\omega) + k(q - \omega) = q^2/3$. The corresponding local equilibrium spectrum reads $n^{\mathrm{loc}}(\omega) = J_\omega/2 + [1 + \sqrt{1 + \lambda^2 J_\omega^2/4}]/\lambda$. Breakdown of energy equipartition manifests itself in a striking way since $n^{\mathrm{loc}}(\omega)$ exhibits a constant pedestal [Fig. 1(b)].

The kinetic theory developed above is valid in the weakly nonlinear regime, i.e., $U/E \ll 1$, $U = H - E$ being the nonlinear contribution to the energy. In the following experimental and numerical study, we show that anomalous thermalization is a robust phenomenon whose signatures are preserved beyond the kinetic regime, i.e., $U > E$. We have designed an experiment in which left- and right-handed circularly polarized optical waves propagate in an isotropic (short-length ultra-low-birefringence) single-mode optical fiber (spun fiber, see Fig. 3). Wave interaction over a significant number of nonlinear lengths L_0 requires the use of high optical powers (400 W) which brings our initial condition into the highly nonlinear regime ($U \gg E$). Note that E and U become comparable at large propagation distances [see inset of Fig. 4(d)]. The evolution of the orthogonal polarization components A_1 and A_2 is governed by the vector NLS Eq. (1) and (2) with $\eta = 1$ and $\kappa = 2$ [12]. The partially coherent waves launched into the fiber have different spectral widths. The narrow (A_1) spectrum (~ 0.05 nm FWHM) is delivered by a Q switch Nd:YAG source emitting 40 ns pulses at a rate of 30 kHz and a wavelength of 1064 nm. The wide (A_2) spectrum ($\sim$1.5 nm FWHM) is generated by passing the Nd:YAG pulses through an annex polarization maintaining fiber. To overcome the resulting time delay ($\sim$13 ns) between the pulses A_1 and A_2, a time-resolved measurement of the spectra has been implemented by

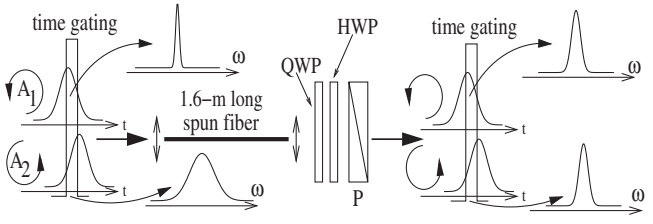

FIG. 3. Schematic representation of the experiment. Left- and right-handed circularly polarized optical waves propagate in the spun fiber and are separated by quarter- and half-wave plates (QWP/HWP) and an optical polarizer (P).

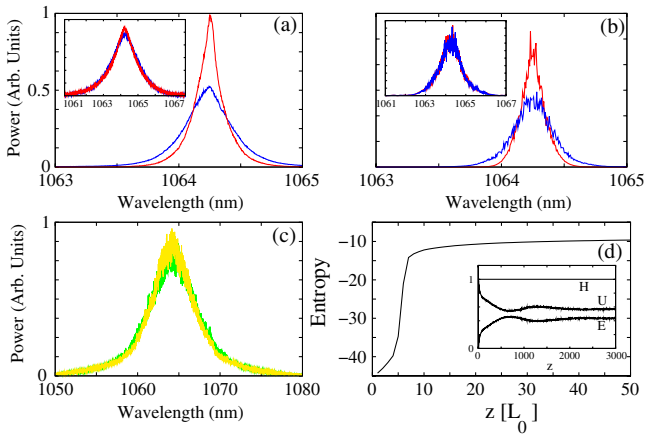

FIG. 4 (color online). (a) Experimental power spectra at the output of the spun fiber without cross-interaction [($\kappa = 0$) in red or gray] and with cross-interaction [($\kappa = 2$) in blue or dark gray]. (b) Corresponding numerical simulations of Eqs. (1) and (2) at $z = 2.5L_0$. (c) Spectra of A_1 and A_2 at $z = 50L_0$ (the two-wave spectra coincide for $z > 15L_0$). (d) Evolution of the entropy and of the kinetic E and nonlinear U energies. Spectra are computed from an ensemble average and by discretizing a wavelength span of 205 nm into 2^{15} points (see procedure of Ref. [18]).

using an acousto-optic time gating technique (see Fig. 3). Short time slices of ~20 ns have been cut into the two pulses: their positions and peak powers have been adjusted to obtain a spectral analysis of the waves carrying an identical average power over the whole time slice.

Because of the presence of the dissipative Raman effect [12], the effective interaction length is considerably reduced, so that only the transient regime of the anomalous thermalization is experimentally accessible. We clearly observed in the experiment that the narrow spectrum A_1 undergoes a significant spectral broadening induced by its interaction with A_2 [see Fig. 4(a)]. Conversely, the wider spectrum A_2 remains almost unchanged from its interaction with A_1 [inset of Fig. 4(a)]. As illustrated in Fig. 4(b), the experimental evolutions of the spectra are reproduced in detail by the numerical simulations of Eqs. (1) and (2) which include the propagation into the annex fiber and neglect the Raman effect. The simulations allowed us to explore the dissipationless asymptotic evolution of the waves. They reveal the existence of an irreversible evolution of the fields $A_{1,2}$ towards a stationary equilibrium state, as illustrated by the saturation of the entropy production process [Fig. 4(d)]. This nonlinear relaxation process is reminiscent of the anomalous thermalization: (i) The nonlinear equilibrium state is characterized by two identical spectra [see Fig. 4(c)], as expected from Eq. (6) for $N_1 = N_2$, (ii) the equilibrium spectra reached for $z > 500L_0$ violate the property of energy equipartition.

In summary, a theoretical, numerical, and experimental study revealed the existence of a novel family of equilibrium states, which are of a fundamental different nature than the usual thermodynamic equilibrium state. Given the universality of the NLS and FWI equations, this work finds applications in many branches of nonlinear science [9,10,12–15]. In the context of multicomponent Bose-Einstein condensates, the condition $\eta = 1$ in Eqs. (1) and (2) is automatically satisfied by simply considering a binary mixture of the same atomic species (same masses) with different internal degrees of freedom.

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[1] *The Fermi-Pasta-Ulam Problem: A Status Report*, edited by G. Gallavotti, Lecture Notes in Physics (Springer, New York, 2007).
[2] P. Villain and M. Lewenstein, Phys. Rev. A **62**, 043601 (2000); P. Villain, P. Öhberg, and M. Lewenstein, Phys. Rev. A **63**, 033607 (2001).
[3] E.g., M. Davis, S. Morgan, and K. Burnett, Phys. Rev. Lett. **87**, 160402 (2001); Phys. Rev. A **66**, 053618 (2002).
[4] E. Infeld, Phys. Rev. Lett. **47**, 717 (1981); G. Cappellini and S. Trillo, J. Opt. Soc. Am. B **8**, 824 (1991).
[5] N. Akhmediev and V. Korneev, Theor. Math. Phys. **69**, 1089 (1986).
[6] B. M. Lake *et al.*, J. Fluid Mech. **83**, 49 (1977).
[7] M. Wu and C. E. Patton, Phys. Rev. Lett. **98**, 047202 (2007).
[8] G. V. VanSimaeys, Ph. Emplit, and M. Haelterman, Phys. Rev. Lett. **87**, 033902 (2001); J. Beeckman *et al.*, Opt. Express **15**, 11 185 (2007).
[9] R. Z. Sagdeev, D. A. Usikov, and G. M. Zaslavsky, *Nonlinear Physics* (Harwood Publ., Chur, 1988).
[10] V. E. Zakharov, V. S. L'vov, and G. Falkovich, *Kolmogorov Spectra of Turbulence I* (Springer, Berlin, 1992); S. Dyachenko *et al.*, Physica (Amsterdam) **57D**, 96 (1992); V. Zakharov, F. Dias, and A. Pushkarev, Phys. Rep. **398**, 1 (2004).
[11] C. Connaughton *et al.*, Phys. Rev. Lett. **95**, 263901 (2005); S. Pitois *et al.*, Phys. Rev. Lett. **97**, 033902 (2006); G. Düring, A. Picozzi, and S. Rica, Physica (Amsterdam) **238D**, 1524 (2009).
[12] Y. S. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals* (Academic Press, New York, 2003).
[13] B. K. Som *et al.*, Phys. Lett. A **72**, 111 (1979); C. J. McKinstrie and D. A. Russell, Phys. Rev. Lett. **61**, 2929 (1988).
[14] J. L. Hammack *et al.*, J. Fluid Mech. **532**, 1 (2005); M. Onorato A. R. Osborne, and M. Serio, Phys. Rev. Lett. **96**, 014503 (2006).
[15] L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation* (Oxford Science Publications, Oxford, 2003).
[16] S. Manakov, JETP **65**, 505 (1973); V. E. Zakharov and E. Schulman, Physica (Amsterdam) **4D**, 270 (1982); **29D**, 283 (1988).
[17] A. M. Balk and E. V. Ferapontov, Transl.-Am. Math. Soc., Ser. 2, **182** (1998), in *Nonlinear Waves and Weak Turbulence*, edited by V. E. Zakharov (Am. Math. Soc., Providence, RI, 1998).
[18] B. Barviau, S. Randoux, and P. Suret, Opt. Lett. **31**, 1696 (2006).