Tradeoffs for Reliable Quantum Information Storage in 2D Systems

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We ask whether there are fundamental limits on storing quantum information reliably in a bounded volume of space. To investigate this question, we study quantum error correcting codes specified by geometrically local commuting constraints on a 2D lattice of finite-dimensional quantum particles. For these 2D systems, we derive a tradeoff between the number of encoded qubits k, the distance of the code d, and the number of particles n. It is shown that $kd^2 = O(n)$ where the coefficient in $O(n)$ depends only on the locality of the constraints and dimension of the Hilbert spaces describing individual particles. The analogous tradeoff for the classical information storage is $k\sqrt{d} = O(n)$.

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Understanding the limits imposed on information processing by the laws of physics is a problem of fundamental and practical importance. A variety of hardware-independent limitations on the power of computers arising from thermodynamics, quantum mechanics, and relativity have been identified recently [1–3].

In this Letter we derive a fundamental upper bound on the amount of quantum information that can be stored reliably in a given volume of a 2D space. This bound stems from geometric locality of quantum operations used to detect and correct errors as well as peculiar features of quantum entanglement in 2D systems. We shall model the information storage using the framework of quantum error correcting codes [4]. Specifically, we consider a system of n finite-dimensional quantum particles (qudits) occupying sites of a 2D lattice Λ. For the sake of clarity we shall consider a regular square lattice of size $\sqrt{n} \times \sqrt{n}$ with open boundary conditions, although our results can be easily extended to more general 2D lattices and periodic boundary conditions. We shall focus on codes for which the code space $\mathcal{C}$ spanned by encoded states can be represented as a common eigenspace of geometrically local pairwise commuting [5] projectors $\Pi_1, \ldots, \Pi_m$ such that $\mathcal{C} = \{|\psi\rangle : \Pi_a|\psi\rangle = |\psi\rangle \text{ for all } a\}$. The code space $\mathcal{C}$ can be regarded as the ground-state subspace of a local gapped Hamiltonian

$$H = -\sum_{a=1}^{m} \Pi_a, \qquad \Pi_a \Pi_b = \Pi_b \Pi_a. \tag{1}$$

Such a code is able to encode $k = \log \dim\mathcal{C}$ logical qubits. Let d be the distance of the code [6]. Our main result is an upper bound

$$k \leq \frac{cn}{d^2}. \tag{2}$$

Here c is a constant coefficient that depends only on locality of the projectors defining the code space and dimension of the Hilbert space describing individual particles. The bound Eq. (2) is tight up to a constant factor since 2D surface codes [7] achieve the scaling $kd^2 \sim n$ for any given n and d [8]. The bound Eq. (2) can be put in sharp contrast to the existence of good stabilizer codes [9] for which $k/n \geq c_1$ and $d/n \geq c_2$ for some constants c_1, c_2. Our result implies that the distance of 2D quantum codes with a nonzero rate k/n is upper bounded by a constant independent of n. It also implies that the distance of any 2D quantum code is at most $O(\sqrt{n})$ extending the results of [10] beyond stabilizer codes.

The motivation for our work stems from several sources. First, quantum error correcting codes provide toy models for how topological quantum order can emerge in the ground states of 2D spin systems with short-range interactions. For example, string-net models introduced by Levin and Wen [11] are described by Hamiltonians involving a sum of commuting projectors; see [12]. The ground state of string-net models defined on a torus (or higher genus surface) has topological degeneracy and can be regarded as a code space of a quantum code. Alternatively, the code space can be chosen as an excited subspace corresponding to a particular configuration of excitations (anyons)—the approach adopted by Kitaev in the topological quantum computing scheme [13]. In this case the code distance is proportional to the distance between anyons while the bound Eq. (2) asserts that the number of encoded qubits is at most a constant fraction of the number of anyons.

Second, one can interpret Eq. (2) as a tradeoff between the storage capacity and stability. The issue of stability has to be addressed if one tries to simulate the ideal Hamiltonian Eq. (1) in a lab; see [14–16] for some experimental and theoretical proposals for implementation of topological quantum order models. The best one can hope for is to approximate individual interactions Π_a with some constant precision ϵ independent of the total number of particles n. A natural question is, how does it affect the ground-state degeneracy and the spectral gap above the ground state? It was recognized in [13] that the distance d plays a crucial role in the stability analysis. By definition of the distance, a weak local perturbation to H lifts the degeneracy of the ground state only in the order $\Omega(d)$ of perturbation theory [13]. A quantitative relation between the zero-temperature stability and the distance has been recently established for Hamiltonians composed of commuting projectors in [17]. The authors of [17] proved the existence of a constant threshold value of the precision ϵ below which the ground-state degeneracy is not lifted up to exponentially small errors and the spectral gap does not close provided that the distance d scales as a positive power of n.

Assuming that the distance d is infinite in the thermodynamic limit, the tradeoff Eq. (2) implies that the amount of quantum information stored per unit volume, $k/n \leq c/d^2$, goes to zero in the thermodynamic limit. This suggests a possible connection between our results and the celebrated holographic principle [18] asserting that the amount of information that can be encoded in a volume of space M scales as the area of the boundary of M. In fact, our main technical tool that we call disentangling lemma asserts that for any encoded state (even for the maximally mixed) the reduced state of any finite region M can be regarded as a pure state entangled with the rest of the lattice by a unitary operator acting only on the boundary of M. Thus only those degrees of freedom located near the boundary of M contribute to the informational content of M.

Generalizing our techniques to quantum codes defined on a D-dimensional lattice yields [19]

$$k \leq \frac{cn}{d^\alpha}, \qquad \alpha = \frac{2}{D-1}. \tag{3}$$

It should be emphasized that throughout this Letter the geometric locality of the constraints Π_a is defined using the standard Euclidean geometry [20]. At the same time, the bound Eq. (2) can be violated for non-Euclidean geometry. For example, Ref. [21] constructed surface codes on general planar graphs with a constant rate k/n and the distance $d \sim \log n$; see also [22]. Also, it is known that stabilizer codes with $k = 1$ and $d \sim \sqrt{n}\log n$ can be constructed on triangulations of some 4D Riemannian surfaces; see Theorem 12.4 in Ref. [23].

One can also ask about the analogue of the tradeoff Eq. (2) for classical information storage. In Ref. [19] we prove that any 2D classical code specified by geometrically local constraints obeys the bound

$$k \leq \frac{cn}{\sqrt{d}}. \tag{4}$$

Here c is a constant depending only on the dimension of individual particles and locality of the constraints specifying the code. Using the mapping from 1D cellular automatons to 2D classical codes from Refs. [24,25] we construct a family of codes with $k \sim \sqrt{n}$ and $d \sim n^{0.8}$, which is quite close to saturating the bound Eq. (4).

Definitions and notations.—We shall assume that the locality of the projectors Π_a can be characterized by a constant interaction range w such that the support of any projector Π_a can be covered by a square block of size $w \times w$. Let

$$\Pi = \prod_{a=1}^{m} \Pi_a \tag{5}$$

be the projector on the code space $\mathcal{C}$. A state ρ is called an encoded state if and only if it has support on the code space $\mathcal{C}$; that is, $\Pi\rho = \rho\Pi = \rho$. We shall say that a region $M \subseteq \Lambda$ is correctable if and only if there exists an error correction operation (a trace preserving completely positive map) $\mathcal{R}$ that corrects the erasure of all particles in M; that is, for any encoded state ρ one has

$$\mathcal{R}(\mathrm{Tr}_M \rho) = \rho. \tag{6}$$

By definition of the distance any region of size smaller than d is correctable.

We shall use the notation $\bar{M} = \Lambda \backslash M$ for the complement of a region M. For any region $M \subseteq \Lambda$ and for any fixed state ρ let $S(M) = -\mathrm{Tr}\rho_M \log\rho_M$ be the von Neumann entropy of the reduced density matrix ρ_M. Using techniques from Refs. [26,27] one can easily show that the error correction condition Eq. (6) has the following entropic counterpart.

Fact 1.—If a region M is correctable, then

$$S(M|\bar{M}) = -S(M) \tag{7}$$

for any encoded state ρ. Here $S(M|\bar{M}) = S(M\bar{M}) - S(\bar{M})$ is the entropy of M conditioned on $\bar{M}$.

(The proof can be found in the extended version of this Letter [19].)

We begin by sketching the steps leading up to our main result, the bound in Eq. (2). Let R be the largest integer m

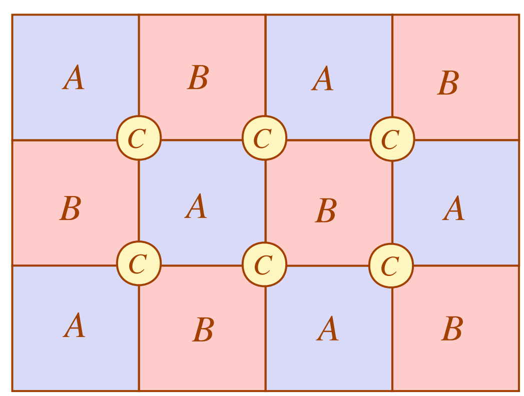

FIG. 1 (color online). The partition of the lattice $\Lambda = ABC$. Each individual block in A and B must be correctable. The region C provides separation between adjacent blocks in A and adjacent blocks in B. It guarantees that the entire regions A and B are correctable. The entropic error correction condition implies that $S(A|BC) = -S(A)$ and $S(B|AC) = -S(B)$ for the maximally mixed encoded state. It yields $k = S(ABC) \leq S(C)$.

such that any square block of size $m \times m$ is correctable. Note that R is at least $\sqrt{d}$ by the definition of the distance.

Consider a partition of the lattice $\Lambda = ABC$ shown in Fig. 1. The regions A and B consist of blocks of size $R \times R$, so that each individual block in A and B is correctable. The total number of blocks is roughly n/R^2. The regions A and B have small corner regions taken out which make up the region C. The purpose of the region C is to provide a sufficiently large separation between the neighboring blocks in A and between the neighboring blocks in B such that any projector Π_a overlaps with at most one block in A and with at most one block in B. It guarantees that the entire regions A and B are correctable (see lemma 2 below). Applying Eq. (7) to regions A and B yields

$$S(A|BC) = -S(A) \quad \text{and} \quad S(B|AC) = -S(B) \qquad (8)$$

for any encoded state. Let ρ be the maximally mixed encoded state such that $k = S(\Lambda)$. Using Eq. (8) we get $S(\Lambda) = S(BC) + S(A|BC) = S(BC) - S(A) \le S(C) + S(B) - S(A)$. Similarly, $S(\Lambda) = S(AC) + S(B|AC) = S(AC) - S(B) \le S(C) + S(A) - S(B)$. Adding together the two bounds yields

$$k = S(\Lambda) \le S(C) \le |C| \sim \frac{n}{R^2}. \qquad (9)$$

The second step in the proof which may be less intuitive is to show that $R \ge cd$ for some constant c depending only on locality of the constraints. In other words, we need to prove that any block of size roughly $d \times d$ is correctable. Our main technical tool will be the disentangling lemma characterizing entanglement properties of the maximally mixed encoded state proportional to the projector on the code space Π. We shall prove that any correctable region M can be completely disentangled from the rest of the lattice by acting only on the boundary of the region (see lemma 1 below). The disentangling operation leaves the region M in a pure state. For any region M let ∂M be the boundary of M, that is, the region covered by the supports of all projectors Π_a that couple M with $\bar{M}$. The following result is a simple corollary of the disentangling lemma.

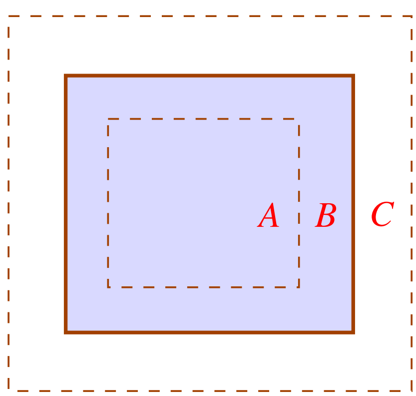

FIG. 2 (color online). Extending the correctability from a region AB to a larger region ABC. The disentangling lemma implies that any encoded state ρ can be represented as $\rho = U_{BC}(\eta_{AB} \otimes \eta_{CD})U_{BC}^\dagger$, where η_{AB} is a pure state independent of ρ. It implies $\mathrm{Tr}_C \eta_{CD} = \rho_D$ and thus $\eta_A \otimes \rho_D = \mathcal{E}(\rho)$, where $\mathcal{E}$ is an "error" erasing the region BC. If BC is correctable, one must be able to reconstruct ρ starting from $\mathcal{E}(\rho)$. Since η_{AB} is known, it means that one can reconstruct ρ starting from ρ_D. Therefore ABC is correctable.

Corollary 1.—Let M be any correctable region. Consider any regions $B \subseteq M$ and $C \subseteq \bar{M}$ such that BC is correctable and $\partial M \subseteq BC$. Then $M \cup C$ is also correctable.

The idea of the proof is illustrated in Fig. 2. Let us apply corollary 1 to a square block M of size $R \times R$. Choose B and C as layers of thickness w adjacent to the surface of M such that $B \subseteq M$ and $C \subseteq \bar{M}$; see Fig. 2. Since all the projectors Π_a have size at most w, the condition $\partial M \subseteq BC$ is satisfied. Note that $|BC| = cwR$ for some constant c. If $|BC| < d$, then BC is correctable and corollary 1 would imply that $M \cup C$ is correctable. But $M \cup C$ is a square block of size larger than R, which contradicts the choice of R. Thus $|BC| \ge d$; that is, $R \ge d/(cw) \sim d$. Substituting this bound into Eq. (9) completes the proof of Eq. (2).

In the rest of the Letter we state and prove the disentangling lemma.

Lemma 1 (disentangling).—Let $M \subseteq \Lambda$ be any correctable region. Suppose that $\partial_+ M = (\partial M) \cap \bar{M}$ is also a correctable region. Then there exists a unitary operator $U_{\partial M}$ acting only on the boundary ∂M such that

$$U_{\partial M} \Pi U_{\partial M}^\dagger = |\phi_M\rangle\langle\phi_M| \otimes \Pi_{\bar{M}}, \qquad (10)$$

for some pure state $|\phi_M\rangle$ and some projector $\Pi_{\bar{M}}$.

It follows from Eq. (10) that $U_{\partial M}$ disentangles any encoded state $|\psi\rangle \in \mathcal{C}$, that is, $U_{\partial M}|\psi\rangle = |\psi_{\rm in}\rangle \otimes |\psi_{\rm out}\rangle$ where $|\psi_{\rm in}\rangle = |\phi_M\rangle$ does not depend on $|\psi\rangle$. Let us also emphasize that lemma 1 holds for any spatial dimension.

Proof of the disentangling lemma.—Consider a partition $\Lambda = ABCD$, where $A = M \backslash \partial M$, $B = M \cap (\partial M)$, $C = \bar{M} \cap (\partial M)$, $D = \bar{M} \backslash \partial M$. By definition, $M = AB$ and $\bar{M} = CD$; see Fig. 2. Using Eq. (5) one can represent Π as a product of commuting projectors acting on MC and CD. The results of Ref. [28] imply that the Hilbert space of C can be decomposed as $\mathcal{H}_C = \bigoplus_x \mathcal{H}_{C'_x} \otimes \mathcal{H}_{C''_x}$ such that $\Pi = \bigoplus_x \Pi^{(x)}_{MC'_x} \otimes \Pi^{(x)}_{C''_x D}$, where $\Pi^{(x)}_{MC'_x}$ and $\Pi^{(x)}_{C''_x D}$ are projectors. Since C is correctable, the direct sum over x contains exactly one term—otherwise it would be possible to distinguish some orthogonal encoded states by measuring x which can be done locally in C. Thus one can subdivide C into two subsystems $C = C'C''$ such that

$$\Pi = U_C(\Pi_{MC'} \otimes \Pi_{C''D})U_C^\dagger. \qquad (11)$$

Using Eq. (5) again one can represent Π as a product of commuting projectors acting on AB and $B\bar{M}$. Applying the same arguments as above one arrives at

$$\Pi = U_B(\Pi_{AB'} \otimes \Pi_{B''\bar{M}})U_B^\dagger, \qquad (12)$$

where $B = B'B''$ is a partition of B into two subsystems and $\Pi_{AB'}$, $\Pi_{B''\bar{M}}$ are some projectors. Define a new projector $\Pi' = U_B^\dagger U_C^\dagger \Pi U_B U_C$. Combining Eqs. (11) and (12) one concludes that $\Pi' = \Pi_{AB'} \otimes \Theta_{B''C'} \otimes \Pi_{C''D}$ for some projector $\Theta_{B''C'}$. The error correction condition Eq. (6) for M implies that $\Pi_{AB'}$ must be one-dimensional, i.e.,

$$\Pi' = |\phi_{AB'}\rangle\langle\phi_{AB'}| \otimes \Theta_{B''C'} \otimes \Pi_{C''D} \tag{13}$$

for some pure state $|\phi_{AB'}\rangle$. As for the projector $\Theta_{B''C'}$, the error correction condition Eq. (6) for M and C (separately) implies that $\Theta_{B''C'}$ is a code space of a code that corrects all errors on B'' and all errors on C'. The no-cloning principle implies that $\Theta_{B''C'}$ must be one-dimensional, that is,

$$\Pi' = |\phi_{AB'}\rangle\langle\phi_{AB'}| \otimes |\phi_{B''C'}\rangle\langle\phi_{B''C'}| \otimes \Pi_{C''D} \tag{14}$$

for some pure state $|\phi_{B''C'}\rangle$. Thus the desired unitary operator $U_{\partial M}$ can be chosen as $U_{\partial M} = W_{B''C'} U_B^\dagger U_C^\dagger$, where $W_{B''C'}$ is an arbitrary unitary operator disentangling the state $|\phi_{B''C'}\rangle$. ■

Our final lemma asserts that the union of two correctable regions M_1 and M_2 that are sufficiently far apart is also correctable.

Lemma 2.—Let $M_1, M_2 \subseteq \Lambda$ be any correctable regions such that any projector Π_a overlaps with at most one of M_1, M_2. Suppose that $\partial_+ M_1$ is also correctable. Then the region $M_1 \cup M_2$ is correctable.

This statement is a simple corollary of the disentangling lemma; see Ref. [19] for the formal proof.

In conclusion, we have shown that there exists a fundamental limit on the amount of quantum information that can be encoded in a given volume of 2D lattice with a given minimal distance. Our work opens up possible connections between the holographic principle and error correction that are worthy of further investigations.

[1] R. Landauer, Nature (London) **335**, 779 (1988).
[2] C. H. Bennett and R. Landauer, Sci. Am. **253**, No. 1, 48 (1985).
[3] S. Lloyd, Nature (London) **406**, 1047 (2000).
[4] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
[5] The commutativity of the constraints specifying a code is a highly desirable property since it allows one to use error correction algorithms based on the syndrome measurement.
[6] Recall that a code has distance d if and only if any measurement performed on any subset of less than d particles reveals no information about an encoded state. Such a code protects encoded information against $[(d-1)/2]$ single-particle errors in the worst-case scenario.
[7] S. Bravyi and A. Kitaev, arXiv:quant-ph/9811052.
[8] Note that the 2D surface code encodes one qubit into approximately d^2 physical qubits; see [7]. In order to encode k qubits one can use k independent copies of the surface code placed next to each other on a plane. It requires roughly kd^2 physical qubits.
[9] A. R. Calderbank and P. W. Shor, Phys. Rev. A **54**, 1098 (1996).
[10] S. Bravyi and B. Terhal, New J. Phys. **11**, 043029 (2009).
[11] M. A. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005).
[12] R. Koenig, B. Reichardt, and G. Vidal, Phys. Rev. B **79**, 195123 (2009).
[13] A. Kitaev, Ann. Phys. (N.Y.) **303**, 2 (2003).
[14] L. Duan, E. Demler, and M. Lukin, Phys. Rev. Lett. **91**, 090402 (2003).
[15] A. Micheli, G. Brennen, and P. Zoller, Nature Phys. **2**, 341 (2006).
[16] R. Koenig, arXiv:0901.1333.
[17] S. Bravyi, M. Hastings, and S. Michalakis, arXiv:1001:0344.
[18] R. Bousso, Rev. Mod. Phys. **74**, 825 (2002).
[19] S. Bravyi, D. Poulin, and B. Terhal, arXiv:0909:5200.
[20] More strictly, our analysis applies to the regular D-dimensional cubic lattice with open or periodic boundary conditions.
[21] G. Zemor, *Coding and Cryptography*, Lecture Notes in Computer Science Vol. 5557 (Springer-Verlag, Berlin, 2009), pp. 259–273.
[22] H. Bombin and M. A. Martin-Delgado, Phys. Rev. A **73**, 062303 (2006).
[23] M. Freedman, D. Meyer, and F. Luo, *Mathematics of Computation* (Chapman and Hall/CRC, Boca Raton, FL, 2002).
[24] D. Roy Chowdhury, S. Basu, I. Sen Gupta, and P. Pal Chaudhuri, IEEE Trans. Comput. **43**, 759 (1994).
[25] M. E. J. Newman and C. Moore, Phys. Rev. E **60**, 5068 (1999).
[26] B. Schumacher and M. A. Nielsen, Phys. Rev. A **54**, 2629 (1996).
[27] P. Hayden, R. Jozsa, D. Petz, and A. Winter, Commun. Math. Phys. **246**, 359 (2004).
[28] S. Bravyi and M. Vyalyi, Quantum Inf. Comput. **5**, 187 (2005).