# Electromagnetic Wave Dynamics in Matter-Wave Superradiant Scattering

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We present a small-signal wave propagation theory on matter-wave superradiant scattering. We show, in a longitudinally excited condensate, that the backward-propagating, superradiantly generated optical field propagates with ultraslow group velocity and that the small-signal gain profile has a Bragg resonance. We further show a unidirectional suppression of optical superradiant scattering, and explain why matter-wave superradiance can occur only when the pump laser is red detuned. This is the first analytical theory on field propagation in matter-wave superradiance that can explain all matter-wave superradiance experiments to date that used a single-frequency, long-pulse, red-detuned laser.

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Coherent matter-wave superradiance is the matter-wave analog of Dicke optical superradiance [1] and it was first observed in an ensemble of Bose-Einstein condensed $^{23}$Na atoms under a pulsed-laser excitation [2]. When a single, far red-detuned, long optical pulse impinges along the short axis of a cigar-shaped Bose-Einstein condensate, it was found that multiple condensates of different momenta formed a highly regular and distinctive unidirectional (with respect to pump laser direction) scattering pattern. It is now understood that these unidirectional, matter-wave superradiant scattering patterns are critically dependent on the geometry and spatial distribution of the condensate.

In a seminal study, Inouye *et al.* [2] explained the observed matter-wave scattering pattern as the result of spontaneous and stimulated Rayleigh superradiant scattering. Three key elements form the core of this Rayleigh superradiant scattering theory [2]: (1) spontaneous and subsequent stimulated Rayleigh scattering along the high gain (the long axis of the condensate) direction, (2) scattered atoms interfere with the local condensate and form a matter-wave grating that further enhances stimulated Rayleigh scattering, and (3) matter-wave amplification.

Following the work of Inouye *et al.* [2], many experimental studies of matter-wave superradiance have lead to the observation of the short-pulsed, bidirectional superradiant effect [3], the Raman superradiant effect [4,5], and matter-wave amplification employing a long-pulse forward superradiant effect to provide a gain medium for an injection-seeded condensate [6–8]. In addition, many theoretical studies [9–14] have also provided substantial insight and understanding of this light-matter-wave interaction process which is of significant importance to the fields of cold atom physics, cold molecular physics, nonlinear optics, and quantum information science.

Despite numerous theoretical efforts several important outstanding questions about the generation and propagation of the electromagnetic field in matter-wave superradiant scattering processes remain unresolved. Two of the most important questions are the following. (1) What are the propagation dynamics of the generated field? (2) Why is there no clear matter-wave superradiant scattering when the laser is detuned to the blue (high energy) side of the one-photon transition? We note that all previous theoretical studies focused on obtaining the gain of the generated field as a function of the pump-field pulse duration (i.e., a rate equation picture). While some propagation properties have been investigated in several numerical studies [13–18], to date there has been no analytical study of the propagation dynamics of the generated light field. In addition, many theoretical studies reported in the literature assume adiabatic elimination of the upper electronic state while taking

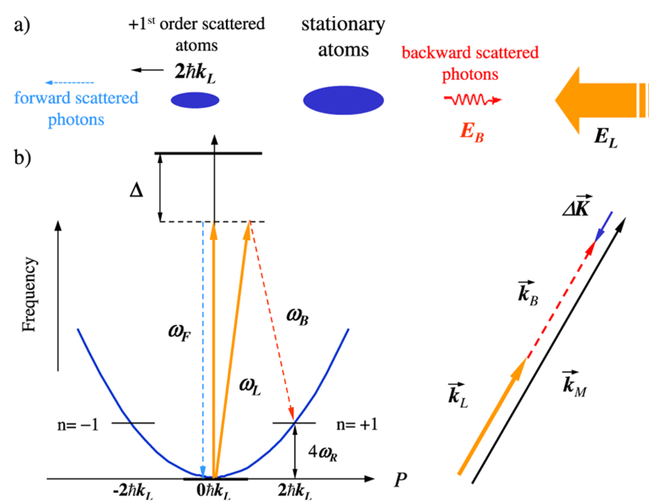


FIG. 1 (color online). (a) Schematic drawing of longitudinally excited matter-wave superradiant scattering of a Bose condensate (only the $n = 0, +1$ orders are shown). We define the scattering in the direction of the pump laser as the positive order of scattering. (b) Schematic drawing of frequency-momentum dispersion and matter-optical wave-vector matching for collective atomic recoil motion (see text).

 

into account a photon relaxation rate $\gamma_{\text{photon}} = c/L \approx 10^{12}$ Hz [13–19]. However, in all experiments reported to date (except one [20]), the one-photon detuning is less than $2 \times 10^{10}$ Hz, and this is inconsistent with any attempt to adiabatically eliminate the upper electronic state using $\gamma_{\text{photon}} \approx 10^{12}$ Hz.

In this Letter, we present a small-signal theory that provides substantial insight into the propagation dynamics of the Rayleigh superradiant-scattering-generated electromagnetic field. In particular, we study a longitudinal excitation geometry where a pump laser pulse impinges and propagates along the long axis of an elongated condensate. In this excitation geometry, we assume that the process begins when spontaneous Rayleigh scattering generates a very weak initial field [21] propagating backward with respect to the pump laser [Fig. 1(a), red wavy arrow. Note that throughout this work backward propagation always means with respect to the pump laser]. We point out that it is essential to include Doppler effects. Indeed, it is the propagation phase-matching and Doppler effects that give rise to the laser detuning effect, the Bragg resonant enhancement of superradiant scattering gain, and ultraslow propagation. Specifically, we show that in the initial growth regime (1) the superradiantly generated field travels backward with respect to the pump field, (2) it travels with an ultraslow group velocity, (3) it has a small-signal gain coefficient only when the matter-optical wave phase matching is achieved by tuning the pump laser to the *red* side of the one-photon electronic transition, and (4) a resonant enhancement of superradiant scattering gain occurs when the frequency difference between the pump and the superradiantly generated fields satisfies a Bragg condition. In addition, the generated field retains full quantum fluctuation characteristics and it grows quadratically as a function of the atom number density. To the best of our knowledge no analytical theory in the literature on field propagation is able to accurately predict the generated optical fields in a matter-wave superradiant scattering process, and none of the effects shown here have been predicted theoretically before.

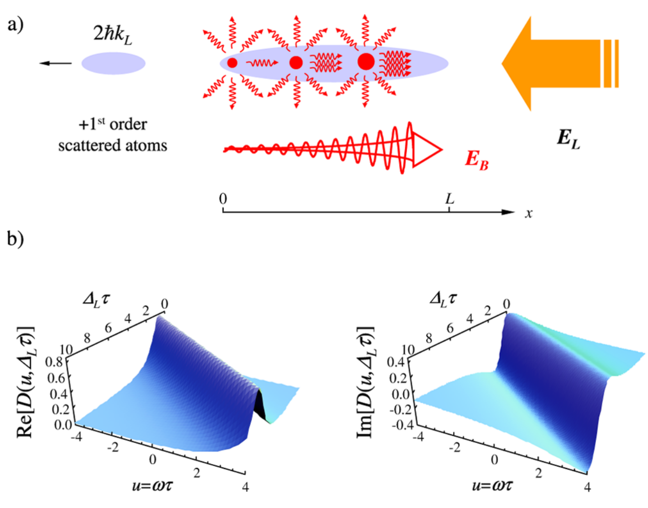


FIG. 2 (color online). (a) Schematic drawing of the growth of the backward-propagating, spontaneously-emitted and stimulated generated photons (the red wavy arrow) by Rayleigh scattering. (b) Real and imaginary parts of the dispersion $D(\omega\tau) = i/[(\omega + 4\omega_R - \Delta_L)\tau + i(4R + \gamma_B)\tau]$ as the function of $u = \omega\tau$ and $\Delta_L\tau$. Left: Bragg enhanced gain. Right: the steep index change responsible for the ultraslow propagation. $\tau = 50$ $\mu$s, $\delta = 5$ GHz, $R = 50$ Hz, $\gamma_B = 24$ kHz.

We begin by investigating the dynamics of a Rayleigh spontaneously-generated weak field propagating backward in a gain medium created by a long optical pump pulse. In our coordinate system the long pump pulse travels in the $-\hat{\mathbf{x}}$ direction. Photons are generated along the path of the pump field by spontaneous processes. However, only photons generated at $x = 0$ (left side of condensate of length $L$) traveling backwards ($+\hat{\mathbf{x}}$ direction) with respect to the pump pulse along the long axis of the condensate will experience maximum gain by the stimulated process as they propagate to $x = L$ [red wavy arrow in Fig. 2(a)]. We focus on the weak-pumping, small-signal growth regime where only the first-order Rayleigh-scattered condensate is important. In this spontaneous and subsequent stimulated emission process, the superradiantly scattered condensate must absorb one photon from the pump laser field $E_L$ ($-\hat{\mathbf{x}}$ direction) and emit one photon to the backward-propagating field $E_B$ ($+\hat{\mathbf{x}}$ direction), acquiring $2\hbar k$ of momentum and moving to the left [Fig. 2(a)]. Processes involving absorption of a laser photon and subsequent emission of a photon in the direction of the pump laser transfer negligible momentum to the condensate and will not be considered here.

We start with the Schrödinger equation describing the coupling of the matter-wave and the electromagnetic fields. After adiabatic elimination of the upper excited electronic state (see discussion on the validity of adiabatic elimination below), the Schrödinger equation for the total system wave function $\Psi = \phi\psi$, where $\phi$ and $\psi$ are the electronic and center-of-motion parts of the total wave function, can be expressed as

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2M}\frac{\partial^2\Psi}{\partial\mathbf{x}^2} + \frac{1}{2}\left[\frac{(\mathbf{d}\cdot\mathbf{E}^{(+)})(\mathbf{d}\cdot\mathbf{E}^{(-)})}{\hbar\Delta} + \text{H.c.}\right]\Psi, \tag{1}$$

where $\mathbf{d}$ is the dipole moment operator and $\mathbf{E}^{(+)} = \mathbf{E}_L^{(+)}e^{-i(\omega_L t + k_L x + k_L\cdot v_D t)} + \mathbf{E}_B^{(+)}e^{-i(\omega_B t - k_B x - k_B\cdot v_D t)}$ with $\mathbf{E}_L^{(+)}$ and $\mathbf{E}_B^{(+)}$ being the amplitudes of the pump and the generated backward-propagating fields. We have explicitly included the one-photon recoil Doppler effect and $v_D = \hbar k_L/M$. In addition, $\Delta = \delta + i\Gamma_0$ where $\delta = \omega_L - \omega_{21}$ is the pump laser detuning to the upper excited electronic state having a spontaneous emission rate of $\Gamma_0$. We emphasize that it is necessary to use the interaction Hamiltonian given in Eq. (1) because of the complex detuning.

Neglecting this term results in an erroneous ac Stark shift and erroneous field equations.

Using Eq. (1) the equation of motion in the interaction picture for the $n$th order scattered atomic mean-field macroscopic wave function component $\psi_n$ is given by

$$\frac{\partial \psi_n}{\partial t} = -(4R + \gamma_B)\psi_n - i g_0 \delta \sum_{\pm} \frac{E_B^{(\pm)}}{E_L^{(\pm)}} \psi_{n\pm1} e^{i(2n\pm1)4\omega_R t \pm i\Delta_L t}, \quad (2)$$

where $\omega_R = \hbar k_L^2/(2M)$ is the one-photon recoil frequency, $g_0 = |\Omega_L|^2/|\Delta|^2$, $R = |\Omega_L|^2\Gamma_0/(4|\Delta|^2)$ with $\Omega_L = d_{12}E_L/\hbar$ being the Rabi frequency of the pump field [22] and $d_{21}$ being the dipole transition matrix element between the ground and the upper excited electronic states.

Three important elements contained in Eq. (2) distinguish it from the macroscopic atomic mean-field wave function equation of motion reported before: (1) $4R$ is the residual of the *complex* ac Stark shift and it describes the rate of excitation; (2) $\gamma_B$ is the Bragg resonance width of the transition between two momentum states coupled by two fields propagating in opposite directions, one from the pump laser and one from the backward-propagating, superradiantly generated field; (3) $\Delta_L = \omega_L - \omega_B$ is the frequency difference between the pump and the superradiantly generated backward-propagating field.

The Maxwell equation obeyed by the amplitude of the generated field $E_B^{(+)}$ is given by

$$\frac{\partial E_B^{(+)}}{\partial x} + \frac{1}{c}\frac{\partial E_B^{(+)}}{\partial t} = -i\frac{\kappa_B}{\Delta} n_0 E_B^{(+)} - i\frac{\kappa_B}{\Delta} E_L^{(+)} \sum_n \psi_n \psi_{n+1}^* e^{i(2n+1)4\omega_R t - i\Delta_L t}, \quad (3)$$

where $\kappa_B = 2\pi\omega_B|d_{12}|^2/(c\hbar)$, $\sum_n |\psi_n|^2 = n_0$ is the atom number density of the initial condensate, and the phase matching condition has been applied (see later discussions).

We now investigate propagation dynamics of the generated field in matter-wave superradiant scattering. In the initial growth regime we neglect the depletion of the ground state condensate (i.e., $\psi_0$ is constant). Correspondingly, all $|n| > 1$ terms are neglected. Taking $n = 1$ in Eq. (2), we obtain the equation of motion for the polarization $P = \psi_0 \psi_1^* e^{i4\omega_R t - i\Delta_L t}$ responsible for generating the backward field (neglecting $n < 0$ components [22])

$$\frac{\partial P}{\partial t} = i(4\omega_R - \Delta_L)P - (4R + \gamma_B)P + i g_0 \delta \frac{E_B^{(+)}}{E_L^{(+)}} n_0. \quad (4)$$

Taking the time Fourier transform of Eqs. (3) and (4) under the phase-matching condition we obtain,

$$\frac{\partial \epsilon(x,\omega)}{\partial x} = \left[-i\frac{\kappa_B}{\Delta}n_0 + i\frac{\omega}{c}\right]\epsilon(x,\omega) + \left[\frac{i\kappa_0 n_0 g_0}{\omega + (4\omega_R - \Delta_L) + i(4R + \gamma_B)}\right]\epsilon(x,\omega), \quad (5)$$

where $\omega$ is the transform variable and $\epsilon(x,\omega)$ is the time Fourier Transform of $E_B^{(+)}(x,t)$. Equation (5) yields

$$\epsilon(x,\omega) = \epsilon(0,\omega)\exp\left[-i\frac{\kappa_B}{\Delta}n_0 x + i\frac{\omega}{c}x\right] \times \exp\left[\frac{i\kappa_0 n_0 g_0 x}{\omega + (4\omega_R - \Delta_L) + i(4R + \gamma_B)}\right]. \quad (6)$$

The first feature to be noticed in Eq. (6) is that when $\Delta_L = \omega_L - \omega_B = 4\omega_R$, corresponding to a Bragg condition, the resonant denominator in the second exponent results in an enhancement of the superradiantly generated field [Fig. 2(b)]. This also suggests that if one injects an initial backward-propagating seed field (in the $+\hat{\mathbf{x}}$ direction at $x = 0$) into the medium and varies the frequency of this seed field, one can map out the matter-wave superradiance gain profile as a function of two-photon detuning [23]. In the following we consider only the case where Bragg resonance enhancement is achieved. It is in this region that significant gain and ultraslow wave propagation can be obtained.

It is very insightful to examine the second exponent in Eq. (6) under the condition of weak, long-pulse excitation where $(4R + \gamma_B)\tau > 4$. In particular we consider a Gaussian-type initial backward field pulse profile $E_{B0}^{(+)}(0,t) = E_{B0}^{(+)}(0,0)e^{-t^2/\tau^2}$ where $E_{B0}^{(+)}(0,0)$ is the amplitude of the backward-propagating field initially generated *stochastically* at $x = 0$ with a pulse length $\tau$, i.e., $\epsilon(0,\omega) = E_{B0}^{(+)}(0,0)\tau\sqrt{\pi}e^{-\omega^2\tau^2/4}$. By Taylor expanding the second exponent to $\omega$-linear terms we obtain, after the inverse Fourier transform,

$$I_B(x,t) = I_B(0,0)e^{2(G-\beta_0)x}\exp\left[-2\left(\frac{t}{\tau} - \frac{x}{V_g\tau}\right)^2\right], \quad (7)$$

where $\beta_0 = \kappa_0 n_0 \Gamma_0/|\Delta|^2$ and $I_B(0,0) \propto |E_{B0}(0,0)|^2$.

Clearly, Eq. (7) describes a field propagating in the $+\hat{\mathbf{x}}$ direction with maximum strength at the end $x = L$. The small-signal Bragg-resonance-enhanced propagation gain constant and ultraslow group velocity [23,24] are

$$G = \frac{\kappa_0 n_0 g_0}{4R + \gamma_B}, \qquad \frac{1}{V_g} = \frac{1}{c} + \frac{G}{4R + \gamma_B}, \quad (8)$$

where $G > \beta_0$ leads to the superradiantly generated field gain threshold $|\Omega_L|^2 > [4R + \gamma_B]\Gamma_0$.

We note that this backward field retains the full quantum fluctuation characteristics prescribed by the initial *stochastic* (spontaneous) process that generates the initial field $E_{B0}^{(+)}(0,0)$, as expected from a superradiance-originated process. It can be shown that when the Taylor expansion is carried to $\omega^2$ terms the present theory predicts a nearly-symmetric pulse shape for the generated field with significant pulse-length spreading. Both features are in excellent agreement with experimental observations for low pump intensities. We further note that Eq. (7) exhibits a quadratic dependence on atom number density, i.e., $I_B(x,t) \propto n_0^2$, a growth characteristic that is expected from a stimulated process such as superradiance.

The predicted ultraslow propagation velocity of the backward-propagating field, which leads to a delayed output, is of critical importance. It is this ultraslow propagation velocity that validates the assumption of adiabatic elimination of the upper electronic state, giving Eq. (1) a solid foundation. We note that in all theoretical treatments reported to date [9–14,19] adiabatic elimination has been assumed, yet a fast photon relaxation rate of $\gamma_{\rm photon} \approx c/L = 3\times 10^{12}$, which stems out of the small size of the condensate (taking $L = 100\ \mu$m), has also been used in several recent studies [13,14,19]. This fast photon damping is inconsistent with the fact that in all (except one [20]) experiments reported to date the one-photon detuning is typically $|\delta| < 2\times 10^{10}$ Hz. Consider a case where $\kappa_0 n_0 \approx 10^{14}/(\text{cm}\cdot\text{s})$ and the pump rate $R = 50$. Equation (8) yields an ultraslow propagation velocity $V_g \approx 60$ cm/s which leads to $\gamma_{\rm photon}^{\rm (slow)} \approx V_g/L \ll |\delta|$, validating adiabatic elimination of the upper electronic state. Correspondingly, the delay time for the generated field to exit the medium of $L = 200\ \mu$m is about $t_D \approx 300\ \mu$s which agrees well with experimental observations [2].

Finally, we examine the matter-optical wave momentum conservation $\Delta K = k_M - (k_L + k_B) \approx 0$ used in deriving Eqs. (2) and (3). For first-order scattering, energy-momentum conservation requires [Fig. 1(b)]

$$4\omega_R + \omega_B - \omega_L = (4\omega_R - 2\omega_L) + \omega_B + \omega_L = 0,$$
$$\Delta K = k_M - n(\omega_B)\frac{\omega_B}{c} - n(\omega_L)\frac{\omega_L}{c} \approx 0.$$

Note that for the matter wave $k_M = 2\omega_L/c$, and the optical wave $k$-vector mismatch per atom is given by $[n(\omega_{B,L}) - 1]\omega_{B,L}/c \approx -\kappa_{B,L}/(N_0\delta)$ [23]; thus, we have

$$4\omega_R + \frac{\kappa(\omega_B) + \kappa(\omega_L)}{N_0\delta} c \approx 0,$$

where $\kappa(\omega_{B,L}) = 2\pi\omega_{B,L}|d_{12}|^2 n_0/c\hbar$ and $N_0$ is the total number of atoms in the condensate. The above relation implies that $\delta < 0$ (a red-detuned pump laser) is required to achieve the collective atom recoil motion expected in a matter-wave superradiant scattering process [25]. We note that this relation agrees with experimental studies [2,3,15].

In conclusion, we have presented a small-signal, matter-wave superradiance theory that can explain all experimental observations reported to date. It shows why matter-wave superradiant scattering can only occur efficiently when the long-pulsed pump laser is red detuned and it predicts a Bragg resonant enhancement in the superradiant gain. We have further shown that matter-wave superradiant scattering generates a field that travels against the pump field with an ultraslow propagation velocity, validating the adiabatic elimination of the upper electronic state. The analytical theory presented here can, for the first time, explain the full propagation dynamics of the matter-wave superradiant scattering process. Although we chose to analyze a longitudinally excited condensate, this theoretical framework is completely general and can be applied to other (e.g., perpendicular) excitation geometries as well.


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