Inverse Scattering and Acousto-Optic Imaging

Guillaume Bal*

Department of Applied Physics and Applied Mathematics, Columbia University, New York, New York 10027, USA

John C. Schotland†

Department of Bioengineering and Graduate Group in Applied Mathematics and Computational Science, University of Pennsylvania, Philadelphia, Pennsylvania 19104, USA

We propose a tomographic method to reconstruct the optical properties of a highly scattering medium from incoherent acousto-optic measurements. The method is based on the solution to an inverse problem for the diffusion equation and makes use of the principle of interior control of boundary measurements by an external wave field.

The acousto-optic effect is a phenomenon in which the optical properties of a material medium are modified due to interaction with acoustic radiation. Brillouin scattering from density fluctuations in a fluid [1] and the ultrasonic modulation of multiply scattered light in a random medium [2] are familiar examples of this effect. It is well known that the scattered optical field carries information about the medium. This principle has been exploited to develop a hybrid imaging modality, known as acousto-optic imaging (AOI), which combines the spectroscopic sensitivity of optical methods with the spatial resolution of ultrasonic imaging [3–17]. AOI holds great promise as a tool to probe hidden structure inside of highly scattering media, such as clouds, paint, and biological tissue—a problem which is of both fundamental interest and considerable applied importance [18]. For instance, in biomedical applications, optical methods provide unique capabilities to assess physiological function including blood volume and tissue oxygenation [19–21]. At the same time, such methods have the molecular selectivity to map gene expression and other markers of biomolecular activity.

In a typical AOI experiment, a highly scattering medium is illuminated by a coherent optical source and the resulting speckle pattern is registered by a detector. A focused ultrasound beam is then introduced and the speckle modulation is recorded as the beam's focus is scanned throughout the medium. Since the scatterers in the medium are displaced by the acoustic wave, the scattered light undergoes a frequency shift which permits the localization of the resulting so-called tagged photons to the volume containing the focus. The intensity images that are obtained in this manner convey information about the medium. However, they are not tomographic, nor are they quantitatively related to the optical properties of the medium.

In this Letter we consider the inverse scattering problem that arises in AOI. We show that it is possible to reconstruct tomographic images of the optical properties of a medium from *incoherent* measurements of multiply scattered light that is modulated by a standing acoustic wave. The prin-

ciple advantages of the proposed method compared to conventional methods for imaging with diffuse light are twofold. (i) The resolution of reconstructed images is, in principle, much higher than in diffuse optical tomography (DOT). In particular, the inverse problem of AOI is well posed and the image resolution is controlled by the acoustic wavelength. In contrast, the inverse scattering problem for diffuse waves is severely ill posed, which leads to the relatively low resolution of DOT [22,23]. Physically, the improvement in resolution in AOI can be understood to be a consequence of controlling an internal degree of freedom of the scattering medium (the density of scatterers) by means of an external wave field. (ii) Neither interferometric measurements of tagged photons nor the use of a focused acoustic wave field is required. This considerably simplifies the experimental realization of the method.

We begin by developing a simple model for the acousto-optic effect in a random medium. The model accounts both for Brillouin scattering and multiple scattering of light due to density fluctuations and is formulated within the framework of radiative transport theory. That is, we do not account for coherent effects in multiple light scattering [24]. Consider the propagation of an acoustic wave in a fluid suspension of particles that scatter and absorb light. If the amplitude of the pressure wave is sufficiently small, each particle will oscillate about its local equilibrium position. We may thus regard the suspension as consisting of independent particles whose number density is spatially modulated due to the presence of the acoustic wave. If we focus our attention on a single neutrally buoyant spherical particle, its velocity $\mathbf{u}$ obeys the equation of motion

$$\rho \frac{d\mathbf{u}}{dt} = \frac{4\pi a \eta}{V} (\mathbf{v} - \mathbf{u}) - \nabla p, \tag{1}$$

where we have not accounted for hydrodynamic interactions between the particles. Here p denotes the pressure, $\mathbf{v}$ is the velocity field in the fluid, a is the radius of the particle, ρ is its mass density, $V = 4\pi a^3/3$, and η is the viscosity of the suspension. Note that the neglect of hydro-

dynamic interactions means that, apart from a transient, the particle moves with the fluid. Consider a standing time-harmonic plane wave of frequency ω with $p = A\cos(\omega t)\cos(\mathbf{k} \cdot \mathbf{r} + \varphi)$, where A is the amplitude of the wave, $\mathbf{k}$ is its wave vector, and φ is the phase [25]. For simplicity, we have assumed that the speed of sound c_s is constant with $k = \omega/c_s$. The corresponding velocity field obeys the linearized Euler equation

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\nabla p. \tag{2}$$

The oscillatory solution to (1) is given by

$$\mathbf{u} = \frac{A}{\rho\omega}\sin(\omega t)\sin(\mathbf{k} \cdot \mathbf{r} + \varphi)\mathbf{k}, \tag{3}$$

and, as a consequence, we obtain for the position of the particle

$$\mathbf{R} = \mathbf{R}_0 - \frac{A}{\rho\omega^2}\cos(\omega t)\sin(\mathbf{k} \cdot \mathbf{r} + \varphi)\mathbf{k}, \tag{4}$$

where $\mathbf{R}_0$ is the equilibrium position.

Let $\mathbf{R}_1, \ldots, \mathbf{R}_N$ denote the positions of the particles in the fluid. Then, since each particle moves independently, we find that

$$\mathbf{R}_i = \mathbf{R}_{0i} - \frac{A}{\rho\omega^2}\cos(\omega t)\sin(\mathbf{k} \cdot \mathbf{r} + \varphi)\mathbf{k}, \quad i = 1, \ldots, N. \tag{5}$$

The number density of particles is defined by $\varrho = \sum_i \delta(\mathbf{r} - \mathbf{R}_i(t))$. Expanding the delta function to first order in the small parameter $\epsilon = A\cos(\omega t)/(\rho c_s^2)$ and using (5), we find that ϱ is given by

$$\varrho(\mathbf{r}) = \varrho_0(\mathbf{r})[1 + \epsilon\cos(\mathbf{k} \cdot \mathbf{r} + \varphi)], \tag{6}$$

where $\varrho_0(\mathbf{r}) = \sum_i \delta(\mathbf{r} - \mathbf{R}_{0i})$ is the equilibrium number density of the particles.

The propagation of multiply scattered light is taken to be described by the diffusion approximation to the radiative transport equation. The electromagnetic energy density u obeys the diffusion equation

$$-\nabla \cdot \left[Dn^2\nabla\left(\frac{u}{n^2}\right)\right] + c\mu_a u = 0 \quad \text{in } \Omega, \tag{7}$$

$$u + \ell\frac{\partial u}{\partial n} = \delta(\mathbf{r} - \mathbf{r}_0) \quad \text{on } \partial\Omega, \tag{8}$$

where n is the index of refraction of the fluid, μ_a is the absorption coefficient, ℓ is the extrapolation length, and $\mathbf{r}_0 \in \partial\Omega$ is the position of a unit-amplitude point source [26,27]. The diffusion coefficient D is defined by

$$D = \frac{c}{3[\mu_a + (1 - g)\mu_s]}, \tag{9}$$

where μ_s is the scattering coefficient and g is the anisotropy of scattering. The scattering and absorption coefficients are related to the number density by $\mu_s = \varrho\sigma_s$ and $\mu_a = \varrho\sigma_a$, where σ_s and σ_a denote the scattering and

absorption cross sections of the particles. We thus see that the optical properties of the medium are modulated by the presence of the acoustic wave. In addition, we account for variations in the index of refraction due to Brillouin scattering according to

$$n(\mathbf{r}) = n_0[1 + \epsilon\gamma\cos(\mathbf{k} \cdot \mathbf{r} + \varphi)], \tag{10}$$

where n_0 is the index of refraction in the absence of the acoustic wave and $\gamma = (\varepsilon - 1)/[3(\varepsilon + 2)]$ is the elasto-optical constant, with ε being the dielectric constant of the fluid [1]. Note that $\gamma \approx 0.3$ in water. Making use of the above definitions, we see that (7) becomes

$$-\nabla \cdot D_\epsilon\nabla\psi_\epsilon + \alpha_\epsilon\psi_\epsilon = 0 \quad \text{in } \Omega, \tag{11}$$

$$\psi_\epsilon + \ell\frac{\partial\psi_\epsilon}{\partial n} = g \quad \text{on } \partial\Omega, \tag{12}$$

where $\psi_\epsilon = u/n^2$, $g = n^2\delta(\mathbf{r} - \mathbf{r}_0)$, and the modified absorption and diffusion coefficients are defined by $\alpha_\epsilon = cn^2\mu_a$ and $D_\epsilon = n^2 D$. Using (6) and (10), we find that to first order in ϵ

$$\alpha_\epsilon(\mathbf{r}) = \alpha_0(\mathbf{r})[1 + \epsilon(2\gamma + 1)\cos(\mathbf{k} \cdot \mathbf{r} + \varphi)], \tag{13}$$

$$D_\epsilon(\mathbf{r}) = D_0(\mathbf{r})[1 + \epsilon(2\gamma - 1)\cos(\mathbf{k} \cdot \mathbf{r} + \varphi)], \tag{14}$$

where α_0 and D_0 are the absorption and diffusion coefficients in the absence of the acoustic wave. The solution to (11) is given by

$$\begin{aligned}
\psi_\epsilon(\mathbf{r}) = \psi_0(\mathbf{r}) - \epsilon\int d^3r'&[(2\gamma + 1)G(\mathbf{r}, \mathbf{r}')\psi_0(\mathbf{r}')\alpha_0(\mathbf{r}') \\
&+ (2\gamma - 1)\nabla_{\mathbf{r}'}G(\mathbf{r}, \mathbf{r}') \cdot \nabla\psi_0(\mathbf{r}')D_0(\mathbf{r}')] \\
&\times \cos(\mathbf{k} \cdot \mathbf{r}' + \varphi) + O(\epsilon^2),
\end{aligned} \tag{15}$$

where the Green function G satisfies

$$-\nabla \cdot D_0(\mathbf{r})\nabla G(\mathbf{r}, \mathbf{r}') + \alpha_0(\mathbf{r})G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \tag{16}$$

and obeys homogeneous boundary conditions of the form (8) with zero right-hand side. We note that ψ_ϵ is proportional to the intensity that is measured by a point detector.

We can estimate the magnitude of the acousto-optic signal from the above analysis. Consider a homogeneous volume of absorption α and diffusion coefficient D. It follows from (17) and (18) that the relative change in intensity due to the presence of an acoustic plane wave is of the order $\Delta I/I \approx \epsilon[1 + (\kappa L)^2]\exp(-\kappa L)$, where L is the source-detector separation and $\kappa = \sqrt{\alpha/D}$. Choosing typical values of the above parameters in tissue, $\kappa = 1$ cm^{-1}, $L = 1$ cm, and $\epsilon = 10^{-3}$, we find that $\Delta I/I \approx 10^{-3}$, which is expected to be observable [28]. We note $\Delta I/I$ would be significantly smaller for the case of a focused beam, assuming equivalent incident power.

The inverse problem of AOI is to reconstruct α_0 and D_0 from ψ_ϵ. To proceed, let us define $\phi = \partial\psi/\partial\epsilon|_{\epsilon=0}$, which can be determined from measurements carried out in the presence and absence of the acoustic wave. Making use of (15) we obtain

$$\phi(\mathbf{r}) = \int K(\mathbf{r}, \mathbf{r}') \cos(\mathbf{k} \cdot \mathbf{r}' + \varphi) d^3 r', \qquad (17)$$

where

$$K(\mathbf{r}, \mathbf{r}') = [(2\gamma + 1)G(\mathbf{r}, \mathbf{r}')\psi_0(\mathbf{r}')\alpha_0(\mathbf{r}')$$
$$+ (2\gamma - 1)\nabla_{\mathbf{r}'}G(\mathbf{r}, \mathbf{r}') \cdot \nabla\psi_0(\mathbf{r}')D_0(\mathbf{r}')]. \quad (18)$$

Suppose we fix the positions of the optical source and detector and vary the wave vector $\mathbf{k}$ and the phase φ. Evidently, it is then possible to recover $K(\mathbf{r}, \mathbf{r}')$ by inversion of a Fourier transform. That is,

$$K(\mathbf{r}, \mathbf{r}') = \int \frac{d^3 k}{(2\pi)^3} e^{-i\mathbf{k} \cdot \mathbf{r}'} [\phi(\mathbf{r}; \mathbf{k}, 0) + i\phi(\mathbf{r}; \mathbf{k}, 3\pi/2)], \tag{19}$$

where the dependence of ϕ on $\mathbf{k}$ and φ has been made explicit. For simplicity, suppose that $\alpha_0 = 0$. Noting that G and ψ_0 depend upon D_0, it follows from (18) that

$$D_0 = \mathcal{A}[D_0], \tag{20}$$

where the *nonlinear* operator $\mathcal{A}$ is defined by

$$\mathcal{A}[D_0](\mathbf{r}') = \frac{K(\mathbf{r}, \mathbf{r}')}{(2\gamma - 1)\nabla_{\mathbf{r}'}G(\mathbf{r}, \mathbf{r}') \cdot \nabla\psi_0(\mathbf{r}')}. \tag{21}$$

Thus D_0 is a fixed point of $\mathcal{A}$, which can be found iteratively according to

$$D_0^{(n+1)} = \mathcal{A}[D_0^{(n)}], \qquad n = 1, 2, \ldots, \tag{22}$$

where $D_0^{(n)} \to D_0$ as $n \to \infty$, provided that $\mathcal{A}$ is contracting. We note that at each step it is necessary to compute the Green function G, which depends upon the current estimate of D_0. We further note that measurements from two independent sources are required to reconstruct both α_0 and D_0.

The above result provides an iterative solution to the inverse problem of AOI. We now describe a *direct* method to solve the inverse problem. We begin by observing that ϕ obeys the equation

$$-\nabla \cdot D_0 \nabla\phi + [(2\gamma - 1)\nabla \cdot D_0 \nabla\psi_0 - (2\gamma + 1)\alpha_0\psi_0]$$
$$\times \cos(\mathbf{k} \cdot \mathbf{r} + \varphi) + \alpha_0\phi = 0 \quad \text{in } \Omega, \quad (23)$$

$$\phi + \ell \frac{\partial\phi}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{24}$$

which follows from (11), (13), and (14). Next we multiply (23) by ψ_0 and (11) by ϕ, take the difference of the two equations that result, and integrate over Ω. After integrating by parts and employing the boundary conditions (8) and (12), we obtain the identity

$$\Sigma(\mathbf{k}, \varphi) = \int_\Omega d^3 r [(2\gamma - 1)D_0(\nabla\psi_0)^2 + (2\gamma + 1)\alpha_0\psi_0^2]$$
$$\times \cos(\mathbf{k} \cdot \mathbf{r} + \varphi), \tag{25}$$

where the surface term Σ is defined by the formula

$$\Sigma(\mathbf{k}, \varphi) = \frac{1}{\ell}D_0(\mathbf{r}_0)[\phi(\mathbf{r}_0) + (2\gamma - 1)\psi_0(\mathbf{r}_0)\cos(\mathbf{k} \cdot \mathbf{r}_0 + \varphi)]$$
$$- \frac{(2\gamma - 1)}{\ell} \int_{\partial\Omega} d^2 r D_0 \psi_0^2 \cos(\mathbf{k} \cdot \mathbf{r} + \varphi). \tag{26}$$

Since ϕ and ψ are known on $\partial\Omega$ from measurements, the function Σ can be determined. Provided that Σ is known for a sufficient number of values of $\mathbf{k}$ and φ, we observe that the Fourier transform of

$$f(\mathbf{r}) = (2\gamma - 1)D_0(\mathbf{r})(\nabla\psi_0(\mathbf{r}))^2 + (2\gamma + 1)\alpha_0(\mathbf{r})\psi_0^2(\mathbf{r}) \tag{27}$$

is known from available measurements. We first consider the case where $\alpha_0 = 0$. We then find that

$$D_0(\mathbf{r}) = \frac{f(\mathbf{r})}{(2\gamma - 1)(\nabla\psi_0(\mathbf{r}))^2}. \tag{28}$$

Making use of (7), we see that ψ_0 obeys the nonlinear equation

$$\nabla \cdot \left[\frac{f}{(\nabla\psi_0)^2} \nabla\psi_0 \right] = 0 \quad \text{in } \Omega, \tag{29}$$

$$\psi_0 + \ell \frac{\partial\psi_0}{\partial n} = g \quad \text{on } \partial\Omega. \tag{30}$$

This equation admits a solution if f is smooth and bounded from below by a positive constant. Once ψ_0 is found by solving (29), we can recover the diffusion coefficient D_0 from (28).

Next we consider the general problem of recovering both α_0 and D_0. In this case, we require data from two sources g_1 and g_2, as specified by the boundary condition (12). We denote the corresponding solutions to (11) by ψ_1 and ψ_2. We also define

$$f_k(\mathbf{r}) = (2\gamma - 1)D_0(\mathbf{r})(\nabla\psi_k(\mathbf{r}))^2 + (2\gamma + 1)\alpha_0(\mathbf{r})\psi_k^2(\mathbf{r}),$$
$$k = 1, 2. \quad (31)$$

Solving for α_0 and D_0 we find

$$\alpha_0(\mathbf{r}) = \frac{f_1(\mathbf{r})(\nabla\psi_2(\mathbf{r}))^2 - f_2(\mathbf{r})(\nabla\psi_1(\mathbf{r}))^2}{(2\gamma + 1)[\psi_1^2(\mathbf{r})(\nabla\psi_2(\mathbf{r}))^2 - \psi_2^2(\mathbf{r})(\nabla\psi_1(\mathbf{r}))^2]}, \tag{32}$$

$$D_0(\mathbf{r}) = \frac{f_2(\mathbf{r})\psi_1^2(\mathbf{r}) - f_1(\mathbf{r})\psi_2^2(\mathbf{r})}{(2\gamma - 1)[\psi_1^2(\mathbf{r})(\nabla\psi_2(\mathbf{r}))^2 - \psi_2^2(\mathbf{r})(\nabla\psi_1(\mathbf{r}))^2]}. \tag{33}$$

The ψ_k are then obtained by solving the system of nonlinear equations

$$-\nabla \cdot D_0 \nabla\psi_k + \alpha_0\psi_k = 0 \quad \text{in } \Omega, \tag{34}$$

$$\psi_k + \ell \frac{\partial\psi_k}{\partial n} = g_k \quad \text{on } \partial\Omega, \tag{35}$$

where α_0 and D_0 are defined by (32) and (33), respectively,

and $k = 1, 2$. Once the ψ_k are found, we can then recover α_0 and D_0 from (32) and (33), which yields the solution to the nonlinear inverse problem of AOI.

Several remarks on the above results are necessary. (i) We observe that the inverse problem of AOI is expected to be well posed since (19) yields a band limited approximation to the kernel K by an inverse Fourier transform. In contrast, the linear inverse problem of DOT is severely ill posed, requiring the inversion of a Laplace transform, a problem which has logarithmic stability [29,30]. This ill-posedness is responsible for the relatively low resolution of images in DOT. Characterizing the stability and ill-posedness of the inverse problem for AOI, along with developing numerical methods for (34), will be the subject of future research. (ii) The nonlinear equation (29) is of the form $\nabla \cdot (f|\nabla \psi|^{p-2}\nabla \psi) = 0$, with $p = 0$. The case $p = 0$ was also addressed in [31,32] in a different physical context and solved numerically using an iterative algorithm similar to the fixed point algorithm presented in (22). The case $p = 1$ was studied in [33], where uniqueness results and numerical methods were described. See also related work in [34]. (iii) The diffusion equation (7) is valid when the energy density varies slowly on the scale of the transport mean free path. This condition breaks down when the acoustic wavelength is sufficiently small. It would thus be of some interest to extend the theory we have developed to the transport regime. (iv) In many applications of interest, the speed of sound is not constant. It is an open problem to generalize our results to this case, although for known, sufficiently localized fluctuations in the sound speed, it is possible to replace the standard Fourier transform by a modified Fourier transform [35].

In conclusion, we have developed a tomographic method for acousto-optic imaging. Neither interferometric measurements of tagged photons nor the use of a focused ultrasound beam is required. Our approach is based on the solution to an inverse problem for the diffusion equation with interior control of boundary measurements.

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*gb2030@columbia.edu

†schotland@seas.upenn.edu

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