Generic New Platform for Topological Quantum Computation Using Semiconductor Heterostructures

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We show that a film of a semiconductor in which s-wave superconductivity and Zeeman splitting are induced by the proximity effect, supports zero-energy Majorana fermion modes in the ordinary vortex excitations. Since time-reversal symmetry is explicitly broken, the edge of the film constitutes a chiral Majorana wire. The heterostructure we propose—a semiconducting thin film sandwiched between an s-wave superconductor and a magnetic insulator—is a generic system which can be used as the platform for topological quantum computation by virtue of the existence of non-Abelian Majorana fermions.

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Introduction.—In two spatial dimensions, where permutation and exchange are not necessarily equivalent, particles can have quantum statistics which are strikingly different from the familiar statistics of bosons and fermions. In situations where the many body ground state wave function is a linear combination of states from a degenerate subspace, a pairwise exchange of the particle coordinates can unitarily *rotate* the ground state wave function in the degenerate subspace. In this case, the exchange statistics is given by a multidimensional unitary matrix representation (as opposed to just a phase factor) of the 2D braid group, and the statistics is non-Abelian [1]. It has been proposed that such a system, where the ground state degeneracy is protected by a gap from local perturbations, can be used as a fault-tolerant platform for topological quantum computation (TQC) [2].

Recently, the $\nu = 5/2$ fractional quantum hall (FQH) state at high magnetic fields and at low temperature has been proposed as a topological qubit [2]. This theoretical conjecture, however, awaits experimental verification [3,4]. An equivalent system, in which the ordered state is in the same universality class as the $5/2$ FQH state, is the spinless (spin-polarized) $p_x + ip_y$ superconductor or superfluid [5]. In a finite magnetic field, a vortex excitation in such a superconductor traps a single, nondegenerate, zero-energy bound state. The key to non-Abelian statistics is that the second-quantized operator for this zero-energy state is self-Hermitian, $\gamma^\dagger = \gamma$, rendering γ a Majorana fermion operator. If the constituent fermions have spin, the spin-degeneracy of the zero-energy excitation spoils the non-Abelian statistics. To circumvent this problem in a realistic superconductor such as strontium ruthenate, it has been proposed that the requisite excitations are the exotic half-quantum vortices [6].

Even though quenching the spin degeneracy by either the application of a magnetic field [7] or by using spinless atomic systems [8] is possible in principle, it is practically very difficult. Therefore, it is desirable to have systems whose most natural excitations themselves follow non-Abelian statistics *in spite of* the electrons carrying a spin quantum number. The recent proposal by Fu and Kane [9] points out one such system—the surface of a strong TI in proximity to an s-wave superconductor—which supports a nondegenerate Majorana fermion excitation in the core of an ordinary vortex. In this Letter, we propose a simple generic TQC platform by showing that it is possible to replace the TI with a regular semiconductor film with spin-orbit coupling, provided the time-reversal symmetry is broken by proximity of the film to a magnetic insulator. It is encouraging that the s-wave proximity effect has already been demonstrated in 2D InAs heterostructures which additionally also have a substantial spin-orbit coupling [10]. Thus, the structure we propose is one of the simplest to realize non-Abelian Majorana fermions in the solid-state context.

Theoretical model.—The single-particle effective Hamiltonian H_0 for the conduction band of a spin-orbit coupled semiconductor in contact with a magnetic insulator is given by (we set $\hbar = 1$ henceforth)

$$H_0 = \frac{p^2}{2m^*} - \mu + V_z\sigma_z + \alpha(\vec{\sigma} \times \vec{p}) \cdot \hat{z}. \qquad (1)$$

Here, m^*, V_z, and μ are the conduction-band effective mass of an electron, effective Zeeman coupling induced by proximity to a magnetic insulator (we neglect the direct coupling of the electrons with the magnetic field from the magnetic insulator), and chemical potential, respectively. The coefficient α describes the strength of the Rashba spin-orbit coupling, and σ_α are the Pauli matrices. Despite the similarity in the spin-orbit-coupling terms, H_0 and the Hamiltonian for the TI surface in Ref. [9] differ by the existence of a spin-diagonal kinetic energy term in Eq. (1). Because of the spin-diagonal kinetic energy, there are, in general, two spin-orbit-split Fermi surfaces in the present system, in contrast to the surface of a TI in which an odd number of bands cross the Fermi level [9]. In Eq. (1), for out-of-plane Zeeman coupling such that $|V_z| > |\mu|$, a single band crosses the Fermi level. Thus, analogous to a strong TI surface (but arising from qualitatively different physics), the system has a single Fermi surface, which is suggestive of non-Abelian topological order if s-wave superconductivity is induced in the film.

The proximity-induced superconductivity in the semiconductor can be described by the Hamiltonian,

$$\hat{H}_p = \int d\mathbf{r}\{\Delta_0(\mathbf{r})\hat{c}^\dagger_\uparrow(\mathbf{r})\hat{c}^\dagger_\downarrow(\mathbf{r}) + \text{H.c.}\}, \tag{2}$$

where $\hat{c}^\dagger_\sigma(\mathbf{r})$ are the creation operators for electrons with spin σ and $\Delta_0(\mathbf{r})$ is the proximity-induced gap. The corresponding BdG equations written in Nambu space become,

$$\begin{pmatrix} H_0 & \Delta_0(\mathbf{r}) \\ \Delta_0^*(\mathbf{r}) & -\sigma_y H_0^* \sigma_y \end{pmatrix}\Psi(\mathbf{r}) = E\Psi(\mathbf{r}), \tag{3}$$

where $\Psi(\mathbf{r})$ is the wave function in the Nambu spinor basis, $\Psi(\mathbf{r}) = [u_\uparrow(\mathbf{r}), u_\downarrow(\mathbf{r}), v_\downarrow(\mathbf{r}), -v_\uparrow(\mathbf{r})]^T$. Using the solutions of the BdG equations, one can define Bogoliubov quasiparticle operators as $\hat{\gamma}^\dagger = \int d\mathbf{r}\sum_\sigma u_\sigma(\mathbf{r})\hat{c}^\dagger_\sigma(\mathbf{r}) + v_\sigma(\mathbf{r})\hat{c}_\sigma(\mathbf{r})$. The bulk excitation spectrum of the BdG equations with $\Delta(r) = \Delta_0$ has a gap for nonvanishing spin-orbit coupling.

BdG equations for a vortex.—We now consider the vortex in the heterostructure shown in Fig. 1, and take the vortex-like configuration of the order parameter: $\Delta_0(r, \theta) = \Delta_0(r)e^{\imath\theta}$. Because of the rotational symmetry, the BdG equations can be decoupled into angular momentum channels indexed by m with the corresponding spinor wave function,

$$\Psi_m(r, \theta) = e^{\imath m\theta}[u_\uparrow(r), u_\downarrow(r)e^{i\theta}, v_\downarrow(r)e^{-i\theta}, -v_\uparrow(r)]^T. \tag{4}$$

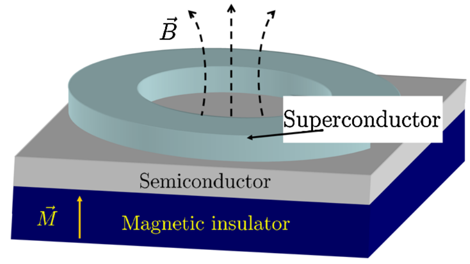

FIG. 1 (color online). Schematic picture of the proposed heterostructure exhibiting Majorana zero-energy bound state inside an ordinary vortex.

Since the BdG equations are particle-hole symmetric, if $\Psi_m(\mathbf{r})$ is a solution with energy E, then $\imath\sigma_y\tau_y\Psi^*_m(\mathbf{r})$ is also a solution at energy $-E$ in the angular momentum channel $-m$. Here, τ_y is defined to be the Pauli matrix in Nambu spinor space. Thus, a zero-energy solution can be nondegenerate only if it exists in the $m = 0$ angular momentum channel.

The radial BdG equations describing the zero-energy state $\Psi(r)$ in the $m = 0$ channel can be written as

$$\begin{pmatrix} H_0 & \Delta_0(r) \\ \Delta_0(r) & -\sigma_y H_0^* \sigma_y \end{pmatrix}\Psi(r) = 0, \qquad H_0 = \begin{pmatrix} -\eta(\partial_r^2 + \frac{1}{r}\partial_r) + V_z - \mu & \alpha(\partial_r + \frac{1}{r}) \\ -\alpha\partial_r & \eta(-\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{r^2}) - V_z - \mu \end{pmatrix} \tag{5}$$

with $\eta = \frac{1}{2m^*}$. Additionally, since the BdG matrix is real, both $\Psi^*(r)$ and $\imath\sigma_y\tau_y\Psi(r)$ are also $E = 0$ solutions. Thus, $\Psi(r)$ can be nondegenerate only if $\imath\sigma_y\tau_y\Psi(r) = \imath\lambda\Psi(r)$, where the $(\imath\sigma_y\tau_y)^2 = -1$ imposes the constraint $\lambda = \pm 1$. Fixing a value of λ allows one to define $\Psi(r)$ in terms of the reduced spinor $\Psi_0(r) = [u_\uparrow(r), u_\downarrow(r)]^T$ using the relations $v_\uparrow(r) = \lambda u_\uparrow(r)$ and $v_\downarrow(r) = \lambda u_\downarrow(r)$. The corresponding reduced BdG equations take the form of a 2×2 matrix differential equation:

$$\begin{pmatrix} -\eta(\partial_r^2 + \frac{1}{r}\partial_r) + V_z - \mu & \lambda\Delta(r) + \alpha(\partial_r + \frac{1}{r}) \\ -\lambda\Delta(r) - \alpha\partial_r & -\eta(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}) - V_z - \mu \end{pmatrix}\Psi_0(r) = 0. \tag{6}$$

We now approximate the radial dependence of $\Delta_0(r)$ by $\Delta_0(r) = 0$ for $r < R$ and $\Delta_0(r) = \Delta_0$ for $r \geq R$. In view of the stability of the putative Majorana zero-energy solution to local changes in the Hamiltonian [5], such an approximation can be made without loss of generality. For $r < R$, the analytical solution of Eq. (6) is given by $\Psi_0(r) = [u_\uparrow J_0(zr), u_\downarrow J_1(zr)]^T$ with the constraint

$$\begin{pmatrix} \eta z^2 + V_z - \mu & z\alpha \\ \alpha z & \eta z^2 - V_z - \mu \end{pmatrix}\begin{pmatrix} u_\uparrow \\ u_\downarrow \end{pmatrix} = 0. \tag{7}$$

Here, $J_n(r)$ are the Bessel functions of the first kind. The characteristic equation for z is

$$(\eta z^2 - \mu)^2 - V_z^2 - \alpha^2 z^2 = 0. \tag{8}$$

Since the Bessel functions are symmetric, we use the roots of Eqn. (8), $\pm z_1$, $\pm z_3$, to find two solutions which are well behaved at the origin: $\Psi_1(r) = [u_\uparrow J_0(z_1 r), u_\downarrow J_1(z_1 r)]^T$ and $\Psi_2(r) = [u_\uparrow J_0(z_3 r), u_\downarrow J_1(z_3 r)]^T$. Therefore, the full solution at $r < R$ is $\Psi_0^<(r) = c_1\Psi_1(r) + c_2\Psi_2(r)$.

At large distances $r > R$, where $\Delta_0(r) = \Delta_0$, the solution to Eq. (6) is complicated. Nevertheless, one can write the solution as a series expansion in $1/r$:

$$\Psi_0(r) = \frac{e^{\imath zr}}{\sqrt{r}} \sum_{n=0,1,2\ldots} \frac{a_n}{r^n} \tag{9}$$

where a_n are the corresponding spinors. The zeroth order coefficient a_0 satisfies the following equation:

$$\begin{pmatrix} \eta z^2 + V_z - \mu & \lambda\Delta_0 + \imath z\alpha \\ -\lambda\Delta_0 - \imath z\alpha & \eta z^2 - V_z - \mu \end{pmatrix} a_0 = 0. \tag{10}$$

The higher order coefficients a_n can be calculated from a_0 using a set of recursion relations [11]. The characteristic equation for Eq. (10) has 4 complex roots for z, which are shown in Fig. 2. Physical solutions of Eq. (6) at $r > R$, $\Psi_0^>(r) = \sum_{n>2} c_n \Psi_n(r)$, require that $\mathrm{Im}[z_n] > 0$. [Here $\Psi_n(r)$ is the solution corresponding to the eigenvalue z_n]. Thus, for $(\mu^2 + \Delta_0^2) > V_z^2$, there are two solutions for $\lambda = \pm 1$. On the other hand, for $(\mu^2 + \Delta_0^2) < V_z^2$, there are three solutions for $\lambda = -1$ and only one for $\lambda = 1$.

To obtain a unique solution for the zero-energy state, the 2-component wave functions $\Psi_0^>(r)$ and $\Psi_0^<(r)$ should satisfy four boundary conditions at $r = R$, namely, the continuity of $\Psi_0(R)$ and $\Psi_0'(R)$. One additional constraint comes from the normalization of the wave function in all space. Thus, there are five independent constraints for the coefficients c_n. A unique zero-energy solution exists if the number of unknown coefficients c_n is five, which is the case for $(\mu^2 + \Delta_0^2) < V_z^2$ and $\lambda = -1$. In this case, the wave functions $\Psi_0^<(r) = \sum_{n=1,2} c_n \Psi_n(r)$ and $\Psi_0^>(r) = \sum_{n=3,4,5} c_n \Psi_n(r)$ have 2 and 3 unknown coefficients, respectively. In all other cases the number of unknowns c_n is smaller than 5, and thus, as we have checked explicitly, solutions for the zero-energy eigenfunction do not exist. From these arguments, one can conclude that, for $(\mu^2 + \Delta_0^2) < V_z^2$, an ordinary vortex in the superconducting condensate contains a unique nondegenerate $E = 0$ solution in the $m = 0$ angular momentum channel. The numerical solution for the zero-energy state is shown in Fig. 2(e). It is straightforward to check that the zero-energy solution corresponds to a self-Hermitian second-quantized operator $\gamma = \gamma^\dagger$: it is a Majorana fermion excitation.

We can also consider the special case with $\eta = 0$, $V_z = 0$ in the above equations, which describes the recent proposal for TQC [9] using zero-energy Majorana bound

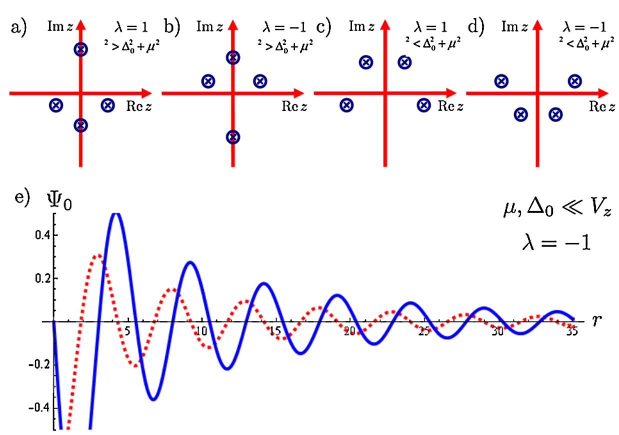

FIG. 2 (color online). Upper panel: complex roots of the Eq. (10) for different values of μ and λ with $\Delta_0 \neq 0$. Lower panel: Numerical solution for the Majorana zero-energy state $\Psi_0(r) = [u(r), v(r)]^T$ for $\lambda = -1$ and $\mu < V_z$. The dashed (red) and solid (blue) lines correspond to $u(r)$ and $v(r)$, respectively. Here we used the following parameters: $\eta = \alpha = V_z = 1$, $\mu = 0$, $\Delta_0 = 0.1$ and $R = 1$. The boundary conditions used are $\Psi_0(0) = (1, 0)^T$ and $\Psi_0(r = 40) = (0, 0)^T$.

states at vortices on the interface of a TI and an s-wave superconductor. In this case, we find a single solution for $r < R$ and a pair of independent solutions for $r > R$. Since the BdG differential equation is now only first order, we need only match the 2-component spinors themselves (derivatives need not match) which yields 3 equations for the 3 coefficients. This leads to a unique Majarana fermion solution at the vortex, which is consistent with Ref. [9]. Interestingly, in contrast to our Hamiltonian for $\eta > 0$, the condition for the existence of a Majorana fermion for $\eta = 0$ is given by $V_z^2 < (\Delta_0^2 + \mu^2)$. The model considered in Ref. [9] and our present system have a similar order parameter structure. In both cases, the order parameter component $\langle c_\uparrow(r) c_\downarrow(r') \rangle$ has an s-wave orbital symmetry while the order parameter components $\langle c_\uparrow(r) c_\uparrow(r') \rangle$ and $\langle c_\downarrow(r) c_\downarrow(r') \rangle$ have $p_x + \imath p_y$ and $p_x - \imath p_y$ orbital symmetries, respectively. On the surface of a TI, because of time-reversal invariance, $|\langle c_\uparrow(r) c_\uparrow(r') \rangle| = |\langle c_\downarrow(r) c_\downarrow(r') \rangle|$. In our system, the ratio of the order parameter components in the two spin sectors is different from 1, and approaches 1 in the limit $\alpha^2/\eta \gg V_z$. In both cases, however, the superconducting pairing potential is s-wave, and is induced by proximity effect. Therefore, the superconducting state and the associated non-Abelian topological character are expected to be robust in the presence of finite disorder.

Topological phase transition.—We have shown above that a nondegenerate Majorana state exists in a vortex in the superconductor only in the parameter regime $(\mu^2 + \Delta_0^2) < V_z^2$. This suggests that there must be a quantum phase transition (QPT) separating the parameter regimes $(\mu^2 + \Delta_0^2) < V_z^2$ and $(\mu^2 + \Delta_0^2) > V_z^2$, even though the system in both regimes is an s-wave superconductor. A nondegenerate zero-energy solution cannot disappear unless a continuum of energy levels appears around $E = 0$. Such a continuum of states at $E = 0$ can only appear if the bulk gap closes, which can be used to define a topological quantum phase transition. In the present system, such a phase transition can be accessed by varying either the Zeeman splitting or the chemical potential. A similar topological quantum phase transition has already been predicted for ultracold atoms with vortices in the spin-orbit coupling [12].

The bulk gap of the present system can be calculated from the bulk excitation spectrum,

$$E^2 = V_z^2 + \Delta_0^2 + \tilde{\epsilon}^2 + \alpha^2 k^2 \pm 2\sqrt{V_z^2 \Delta_0^2 + \tilde{\epsilon}^2 (V_z^2 + \alpha^2 k^2)}, \tag{11}$$

where $\tilde{\epsilon} = \eta k^2 - \mu$. As seen in Fig. 3, the excitation gap first increases as a function of Δ_0 (proximity-induced pair potential) and then decreases and vanishes at a critical point, $\Delta_{0c} = \sqrt{V_z^2 - \mu^2}$, before reopening and increasing with Δ_0. The critical point marks the phase transition between a topologically nontrivial (left) and a topologically trivial (right) s-wave superconducting phases. The scale of the gap in the topologically nontrivial phase is set

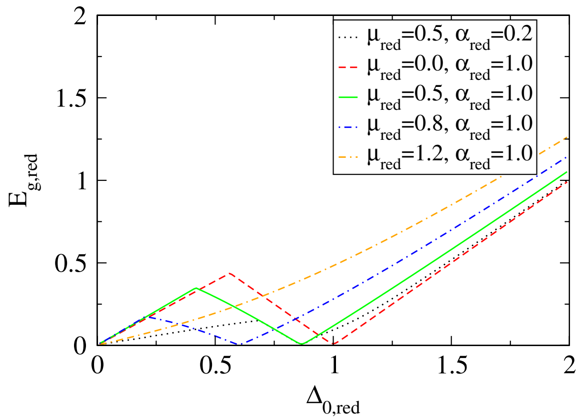

FIG. 3 (color online). Quasiparticle gap versus pairing potential for various values of the chemical potential μ. Here, $E_{g,\text{red}} = E_g/V_z$, $\Delta_{0,\text{red}} = \Delta_0/V_z$, $\mu_{\text{red}} = \mu/V_z$, and $\alpha_{\text{red}} = \alpha/\sqrt{\eta V_z}$. The maximum value of the gap in the topologically nontrivial superconductor, and the corresponding area in the phase diagram, decreases with increasing values of μ. For large negative μ, the system makes a transition to a semiconductor. The phase to the right of the critical point is the topologically trivial s-wave superconductor.

by the strength of the spin-orbit coupling α and the position of the critical point. The fact that the phase on the right-side of the critical point does not support a nondegenerate Majorana mode can be verified by observing that, for these values of Δ_0, it is possible to reduce V_z such that $|V_z| < |\mu|$ without the gap vanishing at any point. This is the phase without Majorana Fermion excitations. In fact, this phase can be reached from the conventional s-wave superconductor with $V_z = 0$ and $\alpha = 0$ without crossing a phase transition.

Majorana edge modes and TQC.—In analogy with Ref. [9], we find that an interface between two superconductor layers, which can be deposited on the semiconductor thin film, supports a pair of zero-energy excitations when the phase difference between the superconductors is π. This geometry can be analyzed in a way that closely follows our derivation of the localized state in a vortex in the $m = 0$ channel, since the BdG Hamiltonian can again be reduced to a real matrix. In this case, we find that, in the parameter regime $(\mu^2 + \Delta_0^2) < V_z^2$, there are 3 linearly independent solutions on each side of the interface. Since the number of constraints to be satisfied at the interface (we assume the interface to be of negligible width) remains 5 as before, one expects a pair of independent zero-energy solutions. The interface, therefore, constitutes a *nonchiral* Majorana wire, which can be exploited for braiding in a way completely analogous to Ref. [9] to perform TQC. Majorana bound states as well as Majorana edge modes in our system can be studied experimentally using nonlocal Andreev reflection [13] and electrically detected interferometry [14,15] experiments.

The experimental implementation of this proposal involves a heterostructure of a magnetic insulator (e.g., EuO), a strong spin-orbit coupled semiconductor (e.g., InAs), and an s-wave superconductor with a large T_c (e.g., Nb). Using these materials, it is possible [11] to induce an effective superconducting pairing potential $\Delta_0 \sim 0.5$ meV and a tunneling-induced effective Zeeman splitting $V_z \sim 1$ meV. Additionally, the strength of spin-orbit interaction α in InAs heterostructures is electric-field tunable and can be made as large as $\alpha \approx 50$ meV Å [16]. With these estimates, the quasiparticle gap E_g is of the order of 1 K. Given that the chemical potential is gate-tunable and can be of the of the order of Δ_0, we numerically estimate the magnitude of the excitation energy for the bound states in a vortex core of size ~20 nm to be of the order of 0.1 K [11], which sets the temperature scale for TQC in this system.

Conclusion.—Our proposed TQC platform should be simpler to implement experimentally than any of the TQC candidates proposed in the literature so far, since it involves a standard heterostructure with a magnetic insulator, a semiconductor film, and an ordinary s-wave superconductor. We believe that the proposed scheme provides the most straightforward method for the solid-state realization of non-Abelian Majorana fermions. A significant practical advantage of the proposed TQC scheme is its generic simplicity: it requires neither special samples or materials nor ultralow temperatures or high magnetic fields.

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