Effect of Polymer Additives on Heat Transport in Turbulent Thermal Convection

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In this Letter, we explore the possible effects of polymer additives on heat transport in turbulent thermal convective flows. Using both direct numerical simulations and shell-model calculations, we show that polymer additives can significantly enhance the heat transport in homogeneous turbulent thermal convection, which mimics the bulk of turbulent Rayleigh-Bénard convection. We also discuss the implication of our results for turbulent Rayleigh-Bénard convection, in which there are boundary layers in addition to the central bulk.

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That adding polymers to turbulent wall-bounded flows can reduce friction drag significantly [1] has long been known, but the effect of polymer additives on heat transport in turbulent thermal convective flows is much less studied. Experimentally, turbulent thermal convection is often investigated in a Rayleigh-Bénard convection cell heated from below and cooled on top. In this Letter, we report our study for homogeneous turbulent thermal convection, which mimics the bulk of turbulent Rayleigh-Bénard convection. Our work shows that polymer additives can significantly enhance the heat transport. We first demonstrate this effect using direct numerical simulations (DNS). Then we study the phenomenon in a shell model of the problem. The usage of this simplified dynamical model allows us to perform a systematic investigation in the parameter space with minimal computational effort. Our predictions, based on the understanding gained by using the shell model, are then checked against and confirmed by the DNS.

Homogeneous turbulent convection has been proposed [2,3] as a three-dimensional convective flow in a box with periodic boundary conditions, driven by a constant temperature gradient along the vertical direction. In the presence of polymers, there is an additional stress tensor that depends on the conformation of the polymers. In Boussinesq approximation [4] and within the finitely extensible nonlinear elastic-Peterlin (FENE-P) model for polymers, we have

$$\frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p + \nu_s \nabla^2 \boldsymbol{u} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{T}} + \alpha g \theta \hat{z} \qquad (1)$$

$$\frac{D\theta}{Dt} = \kappa \nabla^2 \theta + \beta u_z. \qquad (2)$$

Here $\boldsymbol{u}$ and T are the velocity and temperature fields, $D/Dt \equiv \partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$ is the material derivative, p is the pressure, T_0 is the mean temperature, $\theta = T - (T_0 - \beta z)$ is the temperature deviation from a linear profile of gradient $-\beta$, $\hat{z}$ is a unit vector in the vertical direction, and α, ν_s, and κ are, respectively, the volume expansion coefficient, kinematic viscosity and thermal diffusivity of the neat fluid. $\boldsymbol{\mathcal{T}}(\boldsymbol{r}, t) = (\nu_p/\tau)[P(\boldsymbol{r}, t)\boldsymbol{R}(\boldsymbol{r}, t)/\rho_0^2 - 1]$ is the additional stress tensor due to the polymers, where the polymer conformation tensor $\boldsymbol{R}$ is the ensemble average of the product of the end-to-end distance of the polymer chains. The equation of motion for $\boldsymbol{R}$ is:

$$\frac{DR_{\alpha\beta}}{Dt} = \frac{\partial u_\alpha}{\partial r_\gamma} R_{\gamma\beta} + R_{\alpha\gamma}\frac{\partial u_\beta}{\partial r_\gamma} - \frac{1}{\tau}[P(\boldsymbol{r}, t)R_{\alpha\beta} - \rho_0^2 \delta_{\alpha\beta}] \qquad (3)$$

where $P(\boldsymbol{r}, t) = (\rho_m^2 - \rho_0^2)/(\rho_m^2 - R_{\gamma\gamma})$ is the Peterlin function. Here ρ_m^2 and ρ_0^2 are the maximum and the equilibrium values of the trace $R_{\gamma\gamma}$, $\nu_p/\nu_s \sim c$, and τ and c are the relaxation time and volume fraction of the polymers. Equations (1)–(3) are numerically integrated using a standard pseudospectral method in a domain $(2\pi)^3$ with periodic boundary conditions for $\boldsymbol{u}$ and θ, using a spectral code with 144^3 collocation points [5]. Heat transport is measured by the Nusselt number (Nu), which is the ratio of the actual heat transport to that when there was only conduction:

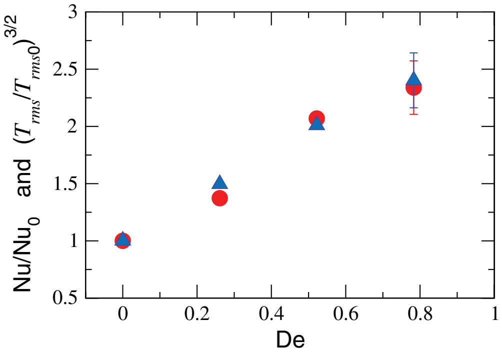

FIG. 1 (color). $\mathrm{Nu}/\mathrm{Nu}_0$ (circles) and $[T_{\mathrm{rms}}/T_{\mathrm{rms0}}]^{3/2}$ (triangles) (see text for definitions) as a function of De obtained in the DNS. The largest error bars are shown.

$$\mathrm{Nu} \equiv \frac{\langle u_z(T-T_0) - \kappa \partial T/\partial z\rangle_A}{\kappa\beta} = \frac{\langle u_z\theta\rangle_V}{\kappa\beta} + 1, \qquad (4)$$

where $\langle\cdots\rangle_A$ denotes an average over (any) horizontal plane of the convection cell and time, and $\langle\cdots\rangle_V$ denotes an average over the whole volume and time. We study how Nu depends on the Deborah number (De), defined by $\mathrm{De} = \tau U_{\mathrm{rms0}}/L \sim \tau\sqrt{g\alpha\beta}$, where U_{rms0} is the rms velocity and L is the energy containing (large) scale in the absence of polymers. As shown in Fig. 1, Nu exceeds Nu_0, the Newtonian value in the absence of polymers at $\mathrm{De} = 0$, for every value of De studied. In particular, there is a twofold increase in Nu for $\mathrm{De} \approx 0.5$. In the case without polymers, it has been known that [6] periodic boundary conditions in the vertical direction introduce exact, exponentially growing (unstable) runaway solutions. In the presence of polymers, it is relatively simple to show that these solutions cannot exist when polymers are stretched. As polymers are stretched when they are acting on the flow, our observed increase of Nu is not due to the effect of any runaway solutions.

To develop a phenomenological understanding of the DNS results, we study the same phenomenon in a shell model, which is a class of simplified dynamical models. By interpreting $R_{\alpha\beta}$ as the dyadic product $B_\alpha B_\beta$ of a vector field $\boldsymbol{B}(\boldsymbol{r}, t)$, a shell model has been developed for homogeneous and isotropic turbulence with polymers [7]. We extend it to include also the temperature field and obtain the shell model for homogeneous turbulent thermal convection with polymers:

$$\dot{u}_n = \frac{i}{3}\left[\Phi_n(u,u) - \frac{\nu_p}{\tau}\Phi_n(B,B)\right] - \nu_s k_n^2 u_n + \alpha g\theta_n \qquad (5)$$

$$\dot{\theta}_n = i(k_{n+1}\theta_{n+1}u_{n+1} + k_n\theta_{n-1}u_n^*) - \kappa k_n^2\theta_n + \beta u_n \qquad (6)$$

$$\dot{B}_n = \frac{i}{3}[\Phi_n(u,B) - \Phi_n(B,u)] - \frac{1}{\tau}B_n - \nu_B k_n^2 B_n \qquad (7)$$

where an overdot denotes a time derivative. The nonlinear interaction terms take the explicit form

$$\begin{aligned}\Phi_n(u,B) = {} & k_n[(1-b)u_{n+2}B_{n+1}^* + (2+b)u_{n+1}^*B_{n+2}] \\ & + k_{n-1}[(2b+1)u_{n-1}^*B_{n+1} - (1-b)u_{n+1}B_{n-1}^*] \\ & + k_{n-2}[(2+b)u_{n-1}B_{n-2} + (2b+1)u_{n-2}B_{n-1}]\end{aligned} \qquad (8)$$

where b is a parameter. Here u_n, B_n, and θ_n are complex variables representing the Fourier amplitudes of $u(k_n)$, $B(k_n)$, and $\theta(k_n)$ with discrete wave vectors $k_n = 2^n k_0$, $n = 0, 1, 2, \ldots$, and $*$ represents the complex conjugate. We take $\nu_B \sim 0$, fix $b = -0.2$ [7], $\nu_p = 1$, $k_0 = 1$, and $\nu_s = \kappa$ such that the Prandtl number (Pr) is one. Also, $\mathrm{Nu} = \mathrm{Re}(\sum_n\langle u_n\theta_n^*\rangle)/(\kappa\beta) + 1$ in analogy to Eq. (4), with $\langle\cdots\rangle$ denoting an average over time.

From Eqs. (5) and (6), we obtain the evolution equations for the kinetic and thermal energies $E_k = \sum_n |u_n|^2/2$ and $E_T = \sum_n|\theta_n|^2/2$:

$$\dot{E}_k = \alpha g \mathrm{Re}\left(\sum_n u_n\theta_n^*\right) - \nu\sum_n k_n^2|u_n|^2 - \epsilon_p \qquad (9)$$

$$\dot{E}_T = \beta \mathrm{Re}\left(\sum_n u_n\theta_n^*\right) - \kappa\sum_n k_n^2|\theta_n|^2 \qquad (10)$$

where $\epsilon_p = \mathrm{Re}[-\frac{i}{3}(\nu_p/\tau)\sum_n u_n^*\Phi_n(B,B)]$ is the rate of energy dissipation due to the polymers and $\epsilon_p = (\nu_p/\tau)\Sigma_n|B_n|^2$ from Eq. (7). Thus in the statistically stationary state,

$$\nu\kappa^2k_0^4(\mathrm{Nu}-1)\mathrm{Ra} = \epsilon_u + \langle\epsilon_p\rangle \equiv \epsilon_{\mathrm{tot}} \qquad (11)$$

$$\kappa\beta^2\mathrm{Nu} = \epsilon_T \qquad (12)$$

where $\epsilon_u = \nu\sum_n k_n^2\langle|u_n|^2\rangle$ and $\epsilon_T = \kappa\sum_n k_n^2\langle|\theta_n|^2\rangle + \kappa\beta^2$ are the mean energy and thermal dissipation rates and $\mathrm{Ra} = \alpha g\beta/(k_0^4\nu\kappa)$ in the shell model.

To proceed, we estimate Nu, ϵ_{tot} (or ϵ_u in the absence of polymers) and ϵ_T in terms of the rms velocity and temperature fluctuations, defined as $U_{\mathrm{rms}} = \sqrt{2\langle E_K\rangle}$ and $T_{\mathrm{rms}} = \sqrt{2\langle E_T\rangle}$. The first estimate is:

$$\mathrm{Re}\sum_n\langle u_n\theta_n^*\rangle = aU_{\mathrm{rms}}T_{\mathrm{rms}} \quad \text{with} \quad a \approx 1. \qquad (13)$$

For the problem in the absence of polymers, we follow [8] and estimate ϵ_u and ϵ_T as

$$\epsilon_u = U_{\mathrm{rms0}}^3/L; \qquad \epsilon_T = U_{\mathrm{rms0}}T_{\mathrm{rms0}}^2/L \qquad (14)$$

where U_{rms0} and T_{rms0} are the rms values in the absence of polymers, and $L \sim 1/k_0$ is the (large) scale of energy and temperature fluctuations. Substituting Eqs. (13) and (14) into Eqs. (11) and (12), and using the definition of Nu, we

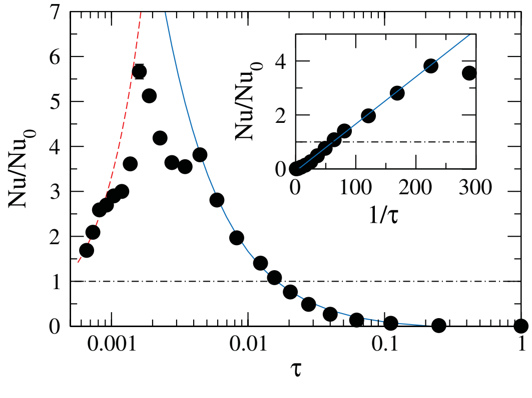

FIG. 2 (color). $\mathrm{Nu}/\mathrm{Nu}_0$ as a function of τ in the shell model with $\alpha g\beta = 1$, $\nu_s = \kappa = 10^{-6}$. The red dashed curve is the $\tau^{3/2}$ fit for small τ and the blue solid curve is the $1/\tau$ fit for large τ. The error bar shown is almost the size of the symbols. Inset: $\mathrm{Nu}/\mathrm{Nu}_0$ as a function of $1/\tau$ to highlight the $1/\tau$ behavior at large τ.

obtain

$$\mathrm{Nu} \sim \sqrt{\mathrm{Pr}\,\mathrm{Ra}} \tag{15}$$

$$U_{\mathrm{rms}0}^2 \sim \alpha g L T_{\mathrm{rms}0}. \tag{16}$$

Equation (15) is the well-known "ultimate-regime" scaling predicted by Kraichnan [9] and obtained in both DNS [8] and shell-model calculations [10] of homogeneous turbulent thermal convection while Eq. (16) is the basic physics underlying this relation.

We now turn to the problem *with polymers*. Figure 2 shows Nu as a function of τ. The Deborah number, computed as $\mathrm{De} \equiv \tau U_{\mathrm{rms}0} k_0$ in the shell model, goes from 0.1 at $\tau = 10^{-3}$ to 10 at $\tau = 0.1$. We clearly see that for not too largeτ, there is a rather large increase in Nu, in agreement with the DNS results (see Fig. 1). For small τ, as the polymers cannot be stretched too much, we expect ϵ_u to be much larger than $\langle \epsilon_p \rangle$. The key observation is that we need to generalize (14) by introducing two different length scales l_0 and l_T:

$$\epsilon_{\mathrm{tot}} = \frac{U_{\mathrm{rms}}^3}{l_0}; \qquad \epsilon_T = \frac{U_{\mathrm{rms}} T_{\mathrm{rms}}^2}{l_T}. \tag{17}$$

Next, we recall that according to our knowledge of the effect of polymers in turbulent flows, the only relevant parameter for the polymers is their relaxation time τ. Thus in general, when polymers are acting, both l_0 and l_T may depend on τ. However, there is no *a priori* reason for l_0 to remain the same as l_T as τ increases. In fact, as shown in Fig. 3, while l_T is definitely a function of τ, l_0 is almost independent of τ for small τ. Using Eqs. (11), (12), and (17), we obtain $U_{\mathrm{rms}}^2 \sim g\alpha\beta l_0 l_T$ and $T_{\mathrm{rms}} \sim \beta l_T$. Thus l_0 being independent of τ for small τ implies that $\langle E_K \rangle \sim \sqrt{\langle E_T \rangle}$, as long as τ is small enough. This further implies that $\mathrm{Nu}/\mathrm{Nu}_0 \sim (T_{\mathrm{rms}}/T_{\mathrm{rms}0})^{3/2}$ for small τ. Our phenomenological considerations are not able to predict any functional form of the dependence of l_T on τ. However, we may guess that the effect of the polymers is to "smooth" out turbulent fluctuations as observed in homogeneous and isotropic turbulence, and tentatively assume that l_T should increase with τ for not too largeτ, which is exactly what Fig. 3 is telling us. For small enough τ, we can assume the increase of l_T with τ to be linear in τ: $l_T \sim \tau$, which leads to $\mathrm{Nu}/\mathrm{Nu}_0 \sim \tau^{3/2}$. This prediction is consistent with our finding in the shell model as shown in Fig. 2.

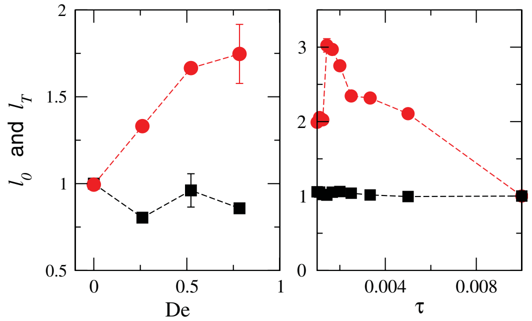

FIG. 3 (color). The length scales l_0 (squares) and l_T (circles), normalized by their corresponding Newtonian values, obtained in DNS (left panel) and in the shell model (right panel). The errors are about 10% in DNS and 3% in the shell model.

When τ becomes very large, most of the energy production should feed polymer stretching. Thus for large τ, $\langle E_K \rangle \sim \sqrt{\langle E_T \rangle}$ no longer holds. We also expect Nu to decrease. If we assume that there exists no singularity in τ such that $\langle E_K \rangle$, $\langle E_T \rangle$ as well as Nu remain as analytic functions of $1/\tau$, then for large τ:

$$\langle E_K \rangle \sim \frac{1}{\tau}; \qquad \langle E_T \rangle \sim \frac{1}{\tau}; \qquad \mathrm{Nu} \sim \sqrt{\langle E_K \rangle \langle E_T \rangle} \sim \frac{1}{\tau}. \tag{18}$$

This argument explains the behavior of Nu for large τ as observed in Fig. 2, and also implies $\langle E_K \rangle \sim \langle E_T \rangle$ for large τ. Both $\langle E_K \rangle \sim \sqrt{\langle E_T \rangle}$ for small τ and $\langle E_K \rangle \sim \langle E_T \rangle$ for large τ, are observed and confirmed in the shell model but are not shown here due to the lack of space.

Let us summarize our phenomenological picture based on the shell model: (1) For small τ, only l_T, but not l_0, depends on τ. l_T should increase with τ and Nu increases as $l_T^{3/2}$. (2) As long as τ is not too large, $\langle E_K \rangle \sim \sqrt{\langle E_T \rangle}$ and $\mathrm{Nu} \sim T_{\mathrm{rms}}^{3/2}$. (3) For very large τ, $\mathrm{Nu} \sim 1/\tau$ and $\langle E_K \rangle \sim \langle E_T \rangle$.

We now go back to check whether our phenomenological picture is supported by the DNS results. In Fig. 3, we compare the behavior of l_T and l_0 computed from DNS to that in the shell model. Indeed l_0 is almost independent of τ while l_T increases with τ for small τ, in good agreement with our finding in the shell model. In Fig. 1, we show, together with $\mathrm{Nu}/\mathrm{Nu}_0$, the behavior of $(T_{\mathrm{rms}}/T_{\mathrm{rms}0})^{3/2}$. It can be seen that our prediction of $\mathrm{Nu} \sim T_{\mathrm{rms}}^{3/2}$ [see point 2 above] is well supported by the DNS results. Hence both points 1 and 2 are well supported by the DNS results. Moreover, our result of $l_T \sim \tau$ for small τ suggests that the typical scale of the temperature fluctuations would

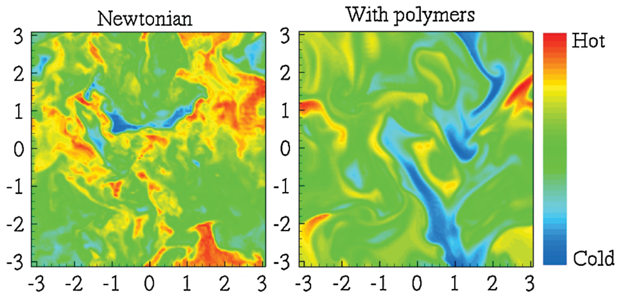

FIG. 4 (color). Two-dimensional cross section of the temperature fluctuation obtained in DNS for Newtonian flow without polymers and with polymers calculated at the largest De.

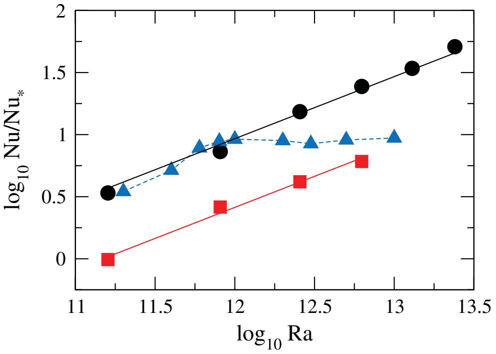

FIG. 5 (color). Nu as a function of Ra in the shell model with Ra increased in two different ways: (i) by decreasing $\nu_s = \kappa$ while keeping β fixed for the case with (circles) and without (squares) polymers, and (ii) by keeping $\nu_s = \kappa$ fixed and increasing β for the case with polymers (triangles).

increase in the presence of polymers. This is in qualitative agreement with the results observed in DNS (see Fig. 4).

The results shown in Fig. 2 have been obtained by keeping Ra constant while varying τ. We have also investigated the Nu(Ra) relationship for fixed τ. There are two possible ways to vary Ra, namely, (i) by varying $\nu_s = \kappa$ while keeping β fixed and (ii) by varying β while keeping $\nu_s = \kappa$ fixed. The results are shown in Fig. 5. In case (i), $\mathrm{Nu} \sim \mathrm{Ra}^{1/2}$ as expected. In case (ii), Nu approaches a constant value for large Ra. This is not surprising since our prediction of $\mathrm{Nu} \sim 1/\tau$ for large τ, at constant Ra, should be properly written in the dimensionless form as $\mathrm{Nu} \sim 1/\mathrm{De}$ where $\mathrm{De} = \tau\sqrt{\alpha g \beta}$. Thus,

$$\mathrm{Nu} \sim \frac{\mathrm{Ra}^{1/2}}{\mathrm{De}} = \left(\frac{\alpha g \beta}{k_0^4 \nu_s \kappa}\right)^{1/2} \frac{1}{\tau\sqrt{\alpha g \beta}} = \frac{1}{\tau k_0^2 \sqrt{\nu_s \kappa}} \qquad (19)$$

which predicts $\mathrm{Nu} \sim \mathrm{const}$ for case (ii) as observed in Fig. 5. This particular result is a different and nontrivial way to test our phenomenological picture. We think that, although we are not able to provide a systematic theory for l_T as a function of τ, the results shown in Figs. 2 and 3 strongly support the phenomenological picture previously described.

Finally, we discuss the implications of our results for turbulent Rayleigh-Bénard convection and thus the possibility to test our findings in experiments. In turbulent Rayleigh-Bénard convection, there are boundary layers in addition to the central bulk. For $\mathrm{Pr} \sim 1$ and $\mathrm{Ra} \sim 10^{11}$–10^{13}, the amount of energy dissipation is approximately 50% due to the bulk and almost 50% to the boundary layers. At such moderate Ra, boundary layers are expected to be laminar and their properties can be investigated using Blasius-type balancing of the constitutive equations [4]. We argue that the addition of polymers in nearly stable boundary layers may produce an increase of friction drag. Increasing drag means increasing momentum flux towards the boundary and consequently, we expect a decrease of Nu. This has indeed been observed in DNS [11] at relatively small Ra. We may think of the difference between Nu, the value with polymers, and Nu_0, the value without polymers, $\delta\mathrm{Nu} \equiv \mathrm{Nu} - \mathrm{Nu}_0$, as being split into two pieces according to the partition of the energy dissipation into its bulk and boundary layer contributions; i.e., we can write $\delta\mathrm{Nu} = \delta\mathrm{Nu}_{\mathrm{BL}} + \delta\mathrm{Nu}_{\mathrm{bulk}}$, where $\delta\mathrm{Nu}_{\mathrm{BL}}$ is the amount of change in Nu corresponding to the energy dissipation in the boundary layers while $\delta\mathrm{Nu}_{\mathrm{bulk}}$ is the amount of change in Nu corresponding to the energy dissipation in the bulk. Our present work shows that $\delta\mathrm{Nu}_{\mathrm{bulk}} > 0$ while our above argument for Blasius boundary layers indicates that $\delta\mathrm{Nu}_{\mathrm{BL}} < 0$. Moreover, it is likely that when Ra is not too large, $|\delta\mathrm{Nu}_{\mathrm{BL}}| \sim \delta\mathrm{Nu}_{\mathrm{bulk}}$ such that polymers do not affect the heat transport significantly. On the other hand, by either (i) increasing Ra, (ii) artificially destroying the contribution to energy dissipation from the boundary layers, or (iii) using a long vertical channel that focusses on the bulk [12], $\delta\mathrm{Nu}_{\mathrm{bulk}}$ should dominate and we should observe an overall increase in Nu in the presence of polymers.

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