Unified Approach to Quantum and Classical Dualities

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We show how classical and quantum dualities, as well as duality relations that appear only in a sector of certain theories (emergent dualities), can be unveiled, and systematically established. Our method relies on the use of morphisms of the bond algebra of a quantum Hamiltonian. Dualities are characterized as unitary mappings implementing such morphisms, whose even powers become symmetries of the quantum problem. Dual variables, which have been guessed in the past, can be derived in our formalism. We obtain new self-dualities for four-dimensional Abelian gauge field theories.

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Introduction.—Dualities appear in nearly all disciplines of physics and play a central role in statistical mechanics and field theory [1,2]. When available, these mathematical transformations provide an elegant, efficient way to obtain information about models that need not be exactly solvable. Most notably, dualities may be used to determine features of phase diagrams such as boundaries between phases, and the exact location of some critical or multicritical points. Historically, dualities were introduced in classical statistical mechanics by Kramers and Wannier (KW) as a relation between the partition function of one system at high temperature (or weak coupling) to the partition function of another (dual) system at low temperatures (or strong coupling). This relation allowed for a determination of the exact critical temperature of the two-dimensional Ising model on a square lattice [3], before the exact solution of the model was available. Later on, it was noticed that, due to the connection between quantum theories in d space dimensions and classical statistical systems in $d+1$ dimensions, dualities can provide relations between quantum theories in the strong coupling and weak coupling regimes [1]. The current work is motivated by a quest for a simple unifying framework for the detection and treatment of dualities.

We will describe an algebraic approach to dualities and self-dualities for systems of arbitrary spatial dimensionality d. We will show that quantum (self-)dualities (a connection between Hamiltonians) become dualities of the related classical statistical problem in $d+1$ dimensions. Thus, quantum and classical (self-)dualities are intrinsically equivalent, yet it will become clear that quantum (self-)dualities are -with the technique presented here- much easier to detect and exploit. The gist of the method is the characterization of quantum (self-)dualities as structure preserving mappings (homomorphisms) between operator algebras which are Hamiltonian dependent. The structure of quantum mechanics further requires that these (self-)duality mappings should be unitarily implementable. In contrast, generalized Jordan-Wigner transformations [4] for example, are dictionaries connecting representations, independent of the structure of any particular Hamiltonian.

Bond algebras and dualities.—Our main thesis is that quantum dualities (self-dualities) are homomorphisms (automorphisms) of bond algebras [5] that preserve locality of interactions and can be implemented through a unitary map. Take a quantum Hamiltonian H, given as a sum of quasilocal operators or bonds $\{h_R\}$ weighed by couplings α_R, $H=\sum_R \alpha_R h_R$. The index R can represent, for example, lattice sites. The bond algebra of H, $\mathcal{A}_H$, is the smallest operator algebra that contains every bond in H, and thus H itself. It can be described as the algebra of all linear combinations of products of bonds $\prod h_R$ and the identity operator. The core idea is that two Hamiltonians H and H_{dual} are dual to each other if there is a unitarily implementable homomorphism Φ between their bond algebras mapping H to H_{dual} up to irrelevant terms in the thermodynamic limit [6]. So we demand that $\Phi(H)=U_D H U_D^\dagger = H_{\text{dual}}+V_B$ where the boundary operator V_B is irrelevant [6]. If H and H_{dual} share the same bonds but with different couplings, then the duality is nothing but a self-duality, established through an automorphism of $\mathcal{A}_H$. This scenario includes the very useful special case of two exchanged couplings representing a weak coupling $\leftrightarrow$ strong coupling exchange. To make clear that this approach is physically sensible, it is enough to notice that such homomorphisms preserve the Heisenberg equations of motion. Notice that the labels $\{R\}$ are completely arbitrary, no reference is made to any particular geometry or dimensionality. The primary algebraic objects are the bonds [5], built out of elementary degrees of freedom such as spins. In the past, quantum dualities such as KW were presented as nonlocal mappings between elementary degrees of freedom. In contrast, duality morphisms are mappings local in the bonds and, remarkably, provide means to derive those nonlocal mappings (which shows that these self-duality automorphisms are indeed the quantum version of the classical order-disorder transformations of Kadanoff and Ceva [7]). That all dualities are manifestations of bond

algebraic morphisms is not obvious. If, however, as is the standard case, two systems are dual to one another on general subsets Λ of an infinite lattice then an exact duality between the two systems exists if and only if the bond algebras of the two systems are identical. The proof of this assertion is straightforward. The proviso of general sublattices implies that a unitary transformation giving rise to the same spectrum may be applied for a general collection of bonds $R \in \Lambda$ and their duals $R' \in \Lambda'$: $U_D H U_D^\dagger = U_D(\sum_{R\in\Lambda} \alpha_R h_R) U_D^\dagger = \sum_{R'\in\Lambda'} \alpha_{R'} h'_{R'} = H_{\text{dual}}$. As this holds for all Λ, it follows that $U_D h_R U_D^\dagger = h'_{R'}$ for all R, R'. If two sets of operators (including the bond operators $\{h_R\}$) are related by a unitary transformation U_D then their algebras are identical. Similarly, if two sets of operators $\{h_R\}$ and $\{h'_{R'}\}$ exhibit an identical algebra then there is a unitary transformation U_D relating them.

In general, self-dualities do not leave H invariant. They are symmetries of the bond algebra $\mathcal{A}_H$, and this is the key to detect them. However, they may become symmetries on appropriate regions of parameter space. If, e.g., U_D exchanges the couplings g and g' in H then at the self-dual point $g = g'$, $[H, U_D] = 0$ (up to the irrelevant terms [6]). Moreover, if U_D effects the exchange for any values of g and g', then for even n, $[U_D^n, H] = 0$ (again, up to irrelevant terms). Taking $n = 2$ we see that

$$\text{self-duality} \rightarrow \sqrt{\text{Quantum Symmetry}}.$$

Thus a self-duality could reveal nontrivial hidden symmetries of a problem. Of course, the symmetries U_D^n, need not be all independent or nontrivial (we will see examples below). One can always add to H an irrelevant boundary term V'_B (related, but not equal to V_B) derived from the bond algebra, so that even for finite systems $[H + V'_B, U_D^n] = 0$ exactly. Thus, it may be useful to work with the more symmetric $H + V'_B$.

As a basic illustration, take $\tilde{H}[j, h] = j\sum_{l=1}^{N-1} \sigma_l^z \sigma_{l+1}^z + h\sum_{l=1}^{N} \sigma_l^x + j\sigma_N^z = H[j, h] + j\sigma_N^z$, $(j\sigma_N^z = V'_B)$ (the σ_l^α are Pauli matrices), where $H[j, h]$ is the Hamiltonian of an Ising chain in a transverse magnetic field (N spins). One can check that $\sigma_1^x \mapsto \sigma_N^z$, $\sigma_N^z \mapsto \sigma_1^x$, $\sigma_i^x \mapsto \sigma_{r(i)}^z \sigma_{r(i)+1}^z$ $(i = 2, 3, \cdots, N)$, $\sigma_i^z \sigma_{i+1}^z \mapsto \sigma_{r(i)}^x$ $(i = 1, \cdots, N-1)$, with $r(i) = N + 1 - i$, gives a unitarily implementable automorphism Φ of the bond algebra of $\tilde{H}$. Φ is clearly a self-duality for the Ising chain $H[j, h]$, $U_D H[j, h] U_D^\dagger = H[h, j] + V_B$, with boundary term $V_B = h\sigma_N^z - j\sigma_1^x$, and it is an exact self-duality for $\tilde{H}$, $\Phi(\tilde{H}[j, h]) = U_D \tilde{H}[j, h] U_D^\dagger = \tilde{H}[h, j]$. In this simple case, $U_D^2 = 1$. The standard approach [8] to this self-duality involves defining nonlocal spin operators—the dual variables—but nothing in principle determines their form; dual variables have to be guessed. In contrast, in our formalism it is natural to use the duality mapping to define dual variables μ_i^α as $\mu_i^\alpha = U_D \sigma_i^\alpha U_D^\dagger$. Then the above relations lead to $\mu_1^x = \sigma_N^z$, $\mu_i^x = \sigma_{r(i)}^z \sigma_{r(i)+1}^z$, $i = 2, \cdots, N$. On the other hand, $\mu_i^z = U_D \sigma_i^z U_D^\dagger = U_D \sigma_i^z \sigma_{i+1}^z \times \cdots \sigma_{N-1}^z \sigma_N^z \times \sigma_N^z U_D^\dagger$, so that, by the duality mapping above, it reduces to $\mu_i^z = \prod_{m=i}^{N} \sigma_{r(m)}^x = \prod_{m=1}^{N+1-i} \sigma_m^x$. Similarly, the Jordan-Wigner dictionary [4] gives rise to a bond algebra mapping when applied to $d = 1$ spin and spinless Fermi systems. The explicit exchange statistics transformation can be derived by solving for one set of bonds in terms of the other. It can be shown that there is no Jordan-Wigner transformation that relates two local Hamiltonians in dimensions $d > 1$: By examining the product of bonds around closed loops an inconsistency is found if local spinless Fermi bilinears could be mapped to local spin terms and vice versa. In the following we disregard boundary terms without further comments.

Dualities and self-dualities in quantum statistical mechanics.—The $d = 3$ orbital compass (OC) model

$$H_{\text{OC}} = -\sum_{\vec{i}} [J_x S_{\vec{i}}^x S_{\vec{i}+\vec{e}_1}^x + J_y S_{\vec{i}}^y S_{\vec{i}+\vec{e}_2}^y + J_z S_{\vec{i}}^z S_{\vec{i}+\vec{e}_3}^z]$$

$(S_{\vec{i}}^\alpha = \frac{1}{2}\sigma_{\vec{i}}^\alpha)$ has been proposed [9] to study orbital ordering in transition metal compounds. A still interesting yet simplified scenario for orbital ordering is provided by the planar OC model (POC)

$$H_{\text{POC}}[J_x, J_y] = -\sum_{\vec{i}} (J_x \sigma_{\vec{i}}^x \sigma_{\vec{i}+\vec{e}_1}^x + J_y \sigma_{\vec{i}}^y \sigma_{\vec{i}+\vec{e}_2}^y) \quad (1)$$

Its bond algebra $\mathcal{A}_{H_{\text{POC}}}$ is generated by $\{\sigma_{\vec{i}}^x \sigma_{\vec{i}+\vec{e}_1}^x, \sigma_{\vec{i}}^y \sigma_{\vec{i}+\vec{e}_2}^y\}_{\vec{i}}$, and it is specified by a few relations: Each bond (i) squares to one, (ii) anticommutes with the four other bonds which share any of its vertices, and (iii) commutes with all other bonds. The mapping $\Phi(\sigma_{\vec{i}}^x \sigma_{\vec{i}+\vec{e}_1}^x) = \sigma_{\vec{i}+\vec{e}_1}^y \sigma_{\vec{i}+\vec{e}_1+\vec{e}_2}^y$, $\Phi(\sigma_{\vec{i}}^y \sigma_{\vec{i}+\vec{e}_2}^y) = \sigma_{\vec{i}+\vec{e}_2}^x \sigma_{\vec{i}+\vec{e}_2+\vec{e}_1}^x$, preserves every relation among bonds, showing a self-duality under $J_x \leftrightarrow J_y$. The POC Hamiltonian is also dual [10] to the Xu-Moore (XM) Hamiltonian [11]

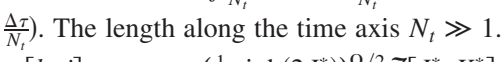
$$H_{\text{XM}}[j, h] = -\sum_{\vec{i}} (j\square\sigma_{\vec{i}}^z + h\sigma_{\vec{i}}^x), \quad (2)$$

(with $\square\sigma_{\vec{i}}^z = \sigma_{\vec{i}}^z \sigma_{\vec{i}+\vec{e}_1}^z \sigma_{\vec{i}+\vec{e}_1+\vec{e}_2}^z \sigma_{\vec{i}+\vec{e}_2}^z$) which was introduced as a simplified model for some aspects of quantum phase transitions in $p + ip$ superconducting arrays. The duality is established through the mapping of bonds $\Phi(\sigma_{\vec{i}}^x \sigma_{\vec{i}+\vec{e}_1}^x) = \square\sigma_{\vec{i}}^z$, $\Phi(\sigma_{\vec{i}}^y \sigma_{\vec{i}+\vec{e}_2}^y) = \sigma_{\vec{i}+\vec{e}_2}^x$, which is indeed given by a unitary U_D. Thus $U_D H_{\text{POC}}[J_x, J_y] U_D^\dagger = H_{\text{XM}}[J_x, J_y]$, and these two models must have the same phase diagram. The quantum-(d)-to-classical-($d + 1$) mapping is much easier for H_{XM} than for H_{POC}, another manifestation of the power of duality transformations and a useful fact if one wants to perform, say, quantum Monte Carlo simulations. The self-duality of the XM Hamiltonian [11] can be deduced from the self-duality of the POC model and the duality just described, or directly as an automorphism of its bond algebra. Applied to the elementary degrees of freedom $\{\sigma_{\vec{i}}^x, \sigma_{\vec{i}}^z\}$, the automorphism returns the nonlocal dual operators of [11].

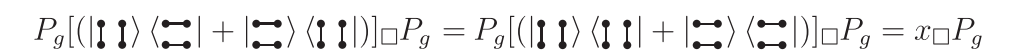
Classical from quantum dualities.—The standard quantum-(d)-to-classical-($d+1$) connection establishes an equivalence between quantum (as unitary mappings) and classical dualities. Take, for example, the XM Hamiltonian $H_{\mathrm{XM}}[j,h]$ of Eq. (2). Its classical rendition is $(\frac{1}{2}\sinh(2J^*))^{\Omega/2}\mathcal{Z}[J,K]$, with $\mathcal{Z}[J,K] \equiv \sum_{[\sigma]} e^{\sum_{\vec{i},t}(J\Box\sigma^z_{\vec{i}}(t)+K\sigma_{\vec{i},t}\sigma_{\vec{i},t+1})}$, $J = j\frac{\Delta\tau}{N_t}$, $J^* = h\frac{\Delta\tau}{N_t}$, and $K = -\frac{1}{2}\ln\tanh(h\frac{\Delta\tau}{N_t})$. The length along the time axis $N_t \gg 1$. Similarly, $H_{\mathrm{XM}}[h,j]$ maps to $(\frac{1}{2}\sinh(2J^*))^{\Omega/2}\mathcal{Z}[J^*,K^*]$, with $K^* = -\frac{1}{2}\ln\tanh(j\frac{\Delta\tau}{N_t})$. It follows already that $\sinh 2J \sinh 2K^* = 1 = \sinh 2J^* \sinh 2K$, yet nothing in principle guarantees any relation between $\mathcal{Z}[J,K]$ and $\mathcal{Z}[J^*,K^*]$ so far. Now, due to the quantum self-duality $H_{\mathrm{XM}}[j,h] = U_D H_{\mathrm{XM}}[h,j]U_D^\dagger$, we have that $\mathrm{Tr}\exp(-\Delta\tau H_{\mathrm{XM}}[j,h]) = \mathrm{Tr}\exp(-\Delta\tau H_{\mathrm{XM}}[h,j])$. Hence $\frac{\mathcal{Z}[J,K]}{(\frac{1}{2}\sinh(2J))^{\Omega/2}} = \frac{\mathcal{Z}[J^*,K^*]}{(\frac{1}{2}\sinh(2J^*))^{\Omega/2}}$, which is indeed the classical self-duality obtained in [11] by considerably more laborious classical methods.

Emergent dualities.—A (self-)duality can emerge in a sector of a theory (e.g., for particular subsets of couplings, or low energy subspace). The projection of a bond algebra onto a sector of the full Hilbert space generates a new bond algebra that may have (self-)dualities not present in the full model. An example is provided by the quantum dimer model (QDM) [12] defined on the orthonormal set of dense dimer coverings of a lattice. The QDM Hamiltonian reads

$$H_{\mathrm{QDM}} = \sum_{\Box} [-t\,(|\,\rangle\langle\,| + |\,\rangle\langle\,|) + v\,(|\,\rangle\langle\,| + |\,\rangle\langle\,|)], \tag{3}$$



with the sum performed over all elementary plaquettes. The QDM contains both a kinetic (t) term that flips one dimer tiling of any plaquette to another (a horizontal covering to a vertical one and vice versa), and a potential (v) term. At the (so-called) RK point $t = v$ [12], the ground states are equal amplitude superpositions of dimer coverings. If P_g is the projection operator onto the ground state sector, then



with $x_\Box = 0$ or 1 on the particular plaquette $\Box$, where $[(|\,\rangle\langle\,| + |\,\rangle\langle\,|)]_\Box$ flips the dimer in the plaquette $\Box$. At the RK point, the projected Hamiltonian becomes $P_g H_{\mathrm{QDM};} P_g = 0$. Since both the kinetic (t) and potential (v) terms are given by $x_\Box P_g$ within the ground state sector, the kinetic and potential operators can be interchanged without affecting the bond algebra. This self-duality emerges exclusively in the ground state sector of the QDM at the RK point.

Dualities in quantum field theory (QFT).—An elementary application of our technique is provided by a free massless scalar field in $1+1$ dimensions [2], with Hamiltonian $H = \frac{1}{2}\int dx[\pi^2(x,t) + (\frac{\partial\phi(x,t)}{\partial x})^2]$, and $[\phi(x,t), \pi(x',t)] = i\delta(x-x')$. (With obvious modifications, this Hamiltonian describes a taut string.) To study this model's bond algebra, it is convenient to discretize it, with lattice spacing a, i.e., $\frac{a}{2}\sum_i[\pi_i^2 + (\phi_{i+1}-\phi_i)^2/a^2]$. The automorphism $\pi_i \mapsto -(\phi_{i+1}-\phi_i)/a$, $(\phi_{i+1}-\phi_i)/a \mapsto \pi_{i+1}$ preserves the canonical commutation relations. The dual variables provide a convenient way to study this self-duality in the continuum. Their discrete form is $\tilde{\phi}_i = a\sum_{m=i+1}^{\infty}\pi_m$, $\tilde{\pi}_i = -(\phi_{i+1}-\phi_i)/a$. Now we can let a go to zero to obtain dual variables in the continuum: $\tilde{\pi}(x,t) = U_D\pi(x,t)U_D^\dagger = -\frac{\partial\phi}{\partial x}(x,t)$, $\tilde{\phi}(x,t) = U_D\phi(x,t)U_D^\dagger = \int_x^\infty dy\,\pi(y,t)$. These are toy examples of solitonic variables. In general, self-dualities can be destroyed by coupling the system to sources, but this is not necessarily the case. Consider the scalar field now coupled to external classical sources A, E: $H^{A,E} = \int dx[\frac{1}{2}\times(\pi-\lambda A)^2 + \frac{1}{2}(\frac{\partial\phi}{\partial x})^2 - \lambda E\phi]$. The self-duality maps $H^{A,E}$ to

$$H^{\tilde{A},\tilde{E}} = \int dx\left[\frac{1}{2}(\pi-\lambda\tilde{A})^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \lambda\tilde{E}\phi + c\,\mathrm{number}\right].$$

The self-duality survives this coupling to external sources, with $\tilde{A}(x,t) = \int_{-\infty}^{x} dw E(w,t)$, $\tilde{E}(x,t) = \frac{\partial A}{\partial x}(x,t)$.

Next we consider $\mathbb{Z}_N$ gauge field theories (GFTs) defined on a Euclidean $(3+1)$-dimensional lattice. The interest in these theories grew out of 't Hooft studies on quark (charge) confinement in pure $SU(N)$ gauge theories [13], that suggest that their most important degrees of freedom near a confinement-deconfinement phase transition are the field configurations taking values in the center subgroup of $SU(N)$, $\mathbb{Z}_N$. To explore this scenario, several author considered Wilson's action for Euclidean lattice GFTs [14], $S = -\frac{1}{g^2}(\sum_\Box \mathrm{Re}\,\mathrm{Tr}(U_{i,j}U_{j,k}U_{k,l}U_{l,i} - 1))$, restricting the fields to take values in $\mathbb{Z}_N$. This is the model we are going to study; thus, $U_{i,j}$ stands for an Nth root of unity attached to the oriented link with endpoints i, j, and $U_{i,j} = U_{j,i}^*$. In the axial gauge the action simplifies

$$S = -\frac{1}{2g^2}\sum_n\sum_{i=1}^{3}[\cos(\theta^i_{n+e_4} - \theta^i_n) + \cos(\Theta^i_n)],$$

($\Theta^1_n = \theta^3_{n+e_2} - \theta^3_n - \theta^2_{n+e_3} + \theta^2_n$, and cyclic permutations thereoff). The goal is to learn about duality properties of amplitudes in QFTs, as given by a path integral over field configurations. Computation of a vacuum to vacuum amplitude $\langle 0\mathrm{out}|0\mathrm{in}\rangle$ amounts to evaluating a partition function. Thus we can apply the bond algebra technique to look for self-dualities in QFTs that are more conveniently quantized through path integrals. To proceed, we need to compute the quantum Hamiltonian equivalent to the gauge

fixed action given above. This is a difficult task for arbitrary N, but the computations were done (in a different context) in [15]. Using these (the coupling K depends on N and g [16])

$$H = -\sum_n \sum_{i=1}^{3} [K V_n^i + \frac{1}{4g^2} \Delta\theta_n^i] + \text{H.c.},$$

where $\Delta\theta_n^3 = U_n^1 U_{n+e_1}^2 U_{n+e_2}^{1\dagger} U_n^{2\dagger}$, and cyclic permutations. There are now $N \times N$ unitary matrices U, V on each link (n, e_i), $i = 1, 2, 3$ of a cubic lattice, (U_n^i, V_n^i denote matrices on the link (n, e_i)). The Us and Vs satisy $(U_n^i)^N = (V_n^i)^N = 1$, $V_n^i U_n^i = \omega U_n^i V_n^i$ ($\omega = e^{2\pi i/N}$), i.e., Weyl's group relations, and matrices on different links commute. $\mathbb{Z}_N$ GFTs have been known for many years to be self-dual for $N = 2$, 3, 4, and it was conjectured that they are no longer self-dual for $N \geq 5$ [14]. We can prove that these theories remain self-dual for all N, as the mapping of bonds

$$\begin{aligned} V_n^1 &\mapsto \Delta\theta_n^1, & \Delta\theta_n^1 &\mapsto V_{n-e_1+e_2+e_3}^{1\dagger} \\ V_n^2 &\mapsto \Delta\theta_{n-e_1+e_2}^2, & \Delta\theta_n^2 &\mapsto V_{n+e_3}^{2\dagger} \\ V_n^3 &\mapsto \Delta\theta_{n-e_1+e_3}^3, & \Delta\theta_n^3 &\mapsto V_{n+e_2}^{3\dagger} \end{aligned} \tag{4}$$

shows. U_D^2 is a new discrete symmetry of this problem, but $U_D^4 = 1$ up to a lattice translation. For large N, these gauge theories are known to display three phases, two of them connected through a confinement-deconfinement phase transition [17]. The self-duality fixes the self-dual coupling g^* at $4g^{*2}K^* = 1$ [16], which gives the exact self-dual coupling for every N (so far only known analytically for $N = 2, 3, 4$). On the other hand, it is shown in [15] (using our approach) that the isotropic $d + 1 = 1 + 1$ N-state vector Potts model has a self-dual point at coupling J^* given by precisely an equivalent relation $2K^* = J^*$. Thus our results explain the puzzling fact [14] that the isotropic classical $d + 1 = 1 + 1$ N-state vector Potts model and the $d + 1 = 3 + 1$ $\mathbb{Z}_N$ GFT share identical self-dual relation: first, both bond algebras (though nonisomorphic) are based on the Weyl algebra, and admit self-duality mappings; and second, both models have quantum couplings satisfying the equation in [16]. The compactness of degrees of freedom (i.e., angular variables), is required for a phase transitions to occur. On one hand, Polyakov [18] showed that compact QED displays no phase transitions in $2 + 1$ dimensions. On the other, we can show that in the limits $N \to \infty$, $a \to 0$, (4) reduces to the well-known self-duality of vacuum QED in $3 + 1$ dimensions $E \mapsto B$, $B \mapsto -E$, which has no phase transitions. We argue that since the self-duality emerges only in $3 + 1$ dimensions, it is important in triggering the phase transitions of these GFTs. So, the presence of both compactness and self-duality is crucial for the existence of a confinement-deconfinement phase transition.

In summary, we developed a unifying and systematic framework for dualities, providing a new perspective to unveil them: (self-)dualities (exact or emergent) can be investigated as homomorphisms of bond algebras. The power of this algebraic approach was exploited to obtain new self-dualities of confining Abelian GFTs in $3 + 1$ dimensions, a new discrete symmetry of these theories, and their self-dual couplings analytically. We prove that the puzzling connections between these GFTs and some confining theories in $1 + 1$ dimensions (vector Potts model) result from these two models having similar algebraic structures and self-dualities. Self-dualities are more easily discovered as automorphisms of bond algebras (quantum) than as relations between partition functions (classical). Furthermore, they can generate otherwise hidden symmetries. Known classical dualities derived in the literature by Fourier transformation [19] can be obtained by our technique. Thus this work hints at a deep connection between operator algebra homomorphisms and the Fourier transform to be at the root of the equivalence between classical and quantum dualities. Our approach to (self-) dualities is applicable to any system (see supplementary material [20]), and clears the way for the development of approximation schemes that preserve these peculiar symmetries.

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