# Third Moments of Conserved Charges as Probes of QCD Phase Structure

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The third moments of conserved charges, the baryon and electric charge numbers, and energy, as well as their mixed moments, carry more information on the state around the QCD phase boundary than previously proposed fluctuation observables and higher order moments. In particular, their signs give plenty of information on the location of the state created in relativistic heavy ion collisions in the temperature and baryon chemical potential plane. We demonstrate this with an effective model.

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Quantum chromodynamics (QCD) is believed to have a rich phase structure in the temperature ($T$) and baryon chemical potential ($\mu_{\rm B}$) plane. Lattice QCD calculations indicate that the chiral and deconfinement phase transitions are a smooth crossover on the temperature axis [1], while various models predict that the phase transition becomes of first order at high density [2]. The existence of the QCD critical point is thus expected. To map these components of the phase diagram on the $T$-$\mu_{\rm B}$ plane is one of the most challenging and stimulating subjects which may be achieved by relativistic heavy ion collisions [3].

Various observables have been proposed for this purpose [4–7]. Most scenarios suggested so far are concerned with fluctuations, such as those of conserved charges, momentum distributions, slope parameters, and so forth. For example, fluctuations of conserved charges behave differently between the hadronic and quark-gluon plasma phases, and may be used as an indicator of the realization of the phase transition [4,5]. The singularity at the critical point, at which the transition is of second order, may also cause enhancements of fluctuations if fireballs created by heavy ion collisions pass near the critical point during the time evolution [6,7]. Because of finite size effects and critical slowing down, however, such singularities are blurred and its experimental confirmation may not be possible [8,9]. In fact, so far no clear evidence for the critical point has been detected in event-by-event analyses [3]. Approaches to use higher order moments for this purpose have been also suggested recently [10] and experimental attempts to measure those higher order moments were reported, for example, in Ref. [11]. Almost all previous studies, however, focus on the *absolute value*, especially the enhancement, of each observable around the phase boundary.

In the present Letter, we propose to employ *signs* of third moments of conserved charges around the averages, which we call, for simplicity, the third moments in the following, to infer the states created by heavy ion collisions. In particular, we consider third moments of conserved quantities, the net baryon and electric charge numbers, and the energy,

$$m_3(ccc) \equiv \frac{\langle(\delta N_c)^3\rangle}{VT^2}, \qquad m_3({\rm EEE}) \equiv \frac{\langle(\delta E)^3\rangle}{VT^5}, \tag{1}$$

where $N_c$ with $c = {\rm B, Q}$ represent the net baryon and electric charge numbers in a subvolume $V$, respectively, $E$ denotes the total energy in $V$, $\delta N_c = N_c - \langle N_c\rangle$, and $\delta E = E - \langle E\rangle$. We also make use of the mixed moments defined as follows:

$$m_3(cc{\rm E}) \equiv \frac{\langle(\delta N_c)^2\delta E\rangle}{VT^3}, \qquad m_3(c{\rm EE}) \equiv \frac{\langle\delta N_c(\delta E)^2\rangle}{VT^4}. \tag{2}$$

To understand the behaviors of these moments around the QCD phase boundary, we first notice that the moments Eqs. (1) and (2) are related to derivatives of the thermodynamic potential per unit volume, $\omega$, up to third order with respect to the corresponding chemical potentials and $T$. The simplest example is $m_3({\rm BBB})$, which is given by

$$m_3({\rm BBB}) = -\frac{\partial^3\omega}{\partial\mu_{\rm B}^3} = \frac{\partial\chi_{\rm B}}{\partial\mu_{\rm B}}, \tag{3}$$

where the baryon number susceptibility, $\chi_{\rm B}$, is defined as

$$\chi_{\rm B} = -\frac{\partial^2\omega}{\partial\mu_{\rm B}^2} = \frac{\langle(\delta N_{\rm B})^2\rangle}{VT}. \tag{4}$$

The baryon number susceptibility $\chi_{\rm B}$ diverges at the critical point and has a peak structure around there [6,12,13]. Since $m_3({\rm BBB})$ is given by the $\mu_{\rm B}$ derivative of $\chi_{\rm B}$ as in Eq. (3), the existence of the peak in $\chi_{\rm B}$ means that $m_3({\rm BBB})$ changes its sign there. Although the precise size and shape of the critical region are not known, various models predict that the peak structure of $\chi_{\rm B}$ well survives far along the crossover line [2,12,14] (see, Fig. 1 as a demonstration of this feature in a simple effective model; the details will be explained later). This means that the near (hadron) and far (quark-gluon) sides of the QCD phase boundary can be distinguished by the sign of $m_3({\rm BBB})$ over a rather wide range around the critical point. It is this

 

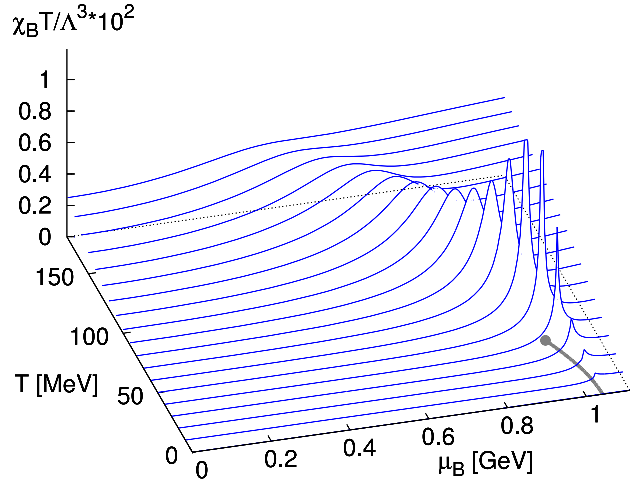


FIG. 1 (color online). $T$ and $\mu_B$ dependence of the baryon number susceptibility $\chi_B$ multiplied by $T$ in the Nambu–Jona-Lasinio model. The bold line on the bottom surface shows the first order phase transition line and the point at the end is the critical point.

feature that third moments carries more information than fluctuations (second moments); fluctuations are, by definition, positive definite and cannot differentiate the near side from the far side as decisively as the third moments. We note that odd power moments around the averages in general do not vanish except the first order one. As we shall see later, all third moments presented in Eqs. (1) and (2) can be expressed in terms of derivatives of corresponding susceptibilities which diverge at the QCD critical point, and hence change their signs there.

The third moments can be measured in heavy ion collisions by the event-by-event analysis similarly to fluctuations, provided that $N_c$ and/or $E$ in a given rapidity range, $\Delta y$, in fireballs created by collisions is determined in each event. The measurement of $N_B$ is difficult because of the difficulty in identifying neutrons. On the other hand, $N_Q$ and $E$ can be measured with the existing experimental techniques. Four out of the seven third moments in Eqs. (1) and (2) composed of $N_Q$ and $E$ thus can be determined experimentally.

All quantities we are considering here, $N_{B,Q}$ and $E$, are conserved charges and the variation of their local densities requires diffusion. In Ref. [4], it was shown that the effect of diffusion is small enough for the fluctuations of the baryon and electric charges if the rapidity range is taken to be $\Delta y \gtrsim 1$ [15]. In the estimate, the one dimensional Bjorken expansion and straight particle trajectories were assumed. If the contraction of hadron phase due to the transverse expansion and the short mean free paths are taken into account, the above estimate will be more relaxed. This conclusion is not altered even if we take the effects of global charge conservation and resonance decays into account [4,5,16]. On the other hand, the experimentally measured charge fluctuations at RHIC [3] are close to those in the hadron phase than the free quark-gluon one. We, however, remark that the values of charge fluctuations in the quark-gluon phase can be similar to that in the hadron phase if the quark and gluons are strongly coupled [17]. The experimental results therefore do not necessarily contradict a realization of the quark-gluon phase.

Once the negativeness of third moments is established experimentally, it is direct evidence of two facts: (1) the existence of a peak structure of corresponding susceptibility in the phase diagram of QCD, and (2) the realization of hot matter beyond the peak, i.e., the quark-gluon plasma, in heavy ion collisions. We emphasize that this statement using the *signs* of third moments does not depend on any specific models. The experimental measurements of signs of moments also have an advantage compared to their absolute values: it is usually essential to normalize experimentally obtained values by extensive observables, such as the total charged particle number $N_{ch}$, in order to compare the experimental results with theoretical predictions [4,5]. In the measurement of signs, however, normalization is not necessary. In the measurement of absolute values, one has to be aware of the effect of global charge conservation when $\Delta y$ is large [16]. It is, however, expected that the effect does not change the signs of the third moments. It is these features that our proposal is less subject to experimental and theoretical ambiguities and more robust than previously proposed ones.

Let us now consider the behavior of third moments other than $m_3(\mathrm{BBB})$ around the critical point. First, the third moment of the net electric charge $m_3(\mathrm{QQQ})$ is calculated to be

$$m_3(\mathrm{QQQ}) = -\frac{\partial^3\omega}{\partial\mu_Q^3} = -\frac{1}{8}\frac{\partial^3\omega}{\partial\mu_B^3} - \frac{3}{8}\frac{\partial^3\omega}{\partial\mu_B^2\mu_I} - \frac{3}{8}\frac{\partial^3\omega}{\partial\mu_B\mu_I^2} - \frac{1}{8}\frac{\partial^3\omega}{\partial\mu_I^3}, \tag{5}$$

where $\mu_Q$ represents the chemical potential associated with $N_Q$, i.e., $\partial/\partial\mu_Q = (2/3)\partial/\partial\mu_u - (1/3)\partial/\partial\mu_d = (\partial/\partial\mu_B + \partial/\partial\mu_I)/2$, and the isospin chemical potential is defined as $\mu_I = (\mu_u - \mu_d)/2$ with $\mu_{u,d}$ being the chemical potentials of the up and down quarks, respectively. In relativistic heavy ion collisions, the effect of isospin symmetry breaking is small. Assuming the isospin symmetry, the second and last terms in the most right-hand side of Eq. (5) vanish, and one obtains

$$m_3(\mathrm{QQQ}) = \frac{1}{8}\frac{\partial}{\partial\mu_B}(\chi_B + 3\chi_I), \tag{6}$$

with the isospin susceptibility $\chi_I = -\partial^2\omega/\partial\mu_I^2$. Under the isospin symmetry, $\chi_I$ does not diverge at the critical point because the critical fluctuation does not couple to the isospin density [7]. The critical behavior of the term in the parenthesis in Eq. (6) in the vicinity of the critical point is thus solely governed by $\chi_B$. Since $m_3(\mathrm{QQQ})$ is a $\mu_B$ derivative of this term, a similar behavior as $m_3(\mathrm{BBB})$ is expected.

Next, it can be shown that mixed moments including a single $E$ are concisely given by

$$m_3(cc\mathrm{E}) = \frac{1}{T}\frac{\partial(T\chi_c)}{\partial T}\bigg|_{\hat{\mu}}, \tag{7}$$

with $c = \mathrm{B}, \mathrm{Q}$, where $\chi_Q \equiv -\partial^2\omega/\partial\mu_Q^2 = (\chi_B + \chi_I)/4$ is the electric charge susceptibility. The $T$ derivative in Eq. (7) is taken along the radial direction from the origin with fixed $\hat{\mu} \equiv \mu_B/T$, i.e., $\partial/\partial T|_{\hat{\mu}} = \partial/\partial T|_{\mu_B} + (\mu_B/T)\partial/\partial\mu_B|_T$. Since $T\chi_c$ diverges at the critical point, Eq. (7) again leads to a similar behavior of $m_3(cc\mathrm{E})$ as the above-mentioned moments.

To argue the behaviors of remaining third moments including two or three $E$'s, it is convenient to first define $C_{\hat{\mu}} = -T(\partial^2\omega/\partial T^2)_{\hat{\mu}} = \langle(\delta E)^2\rangle/VT^2$. The third moments are then given by

$$m_3(\mathrm{EEE}) = \frac{1}{T^3}\frac{\partial(T^2 C_{\hat{\mu}})}{\partial T}\bigg|_{\hat{\mu}}, \tag{8}$$

$$m_3(\mathrm{BEE}) = 2m_3(\mathrm{QEE}) = \frac{1}{T}\frac{\partial C_{\hat{\mu}}}{\partial \mu_B}. \tag{9}$$

Since $C_{\hat{\mu}}$ is the second derivative of $\omega$ along the radial direction, it diverges at the critical point which belongs to the same universality class as that of the 3D Ising model. Therefore, $m_3(\mathrm{EEE})$, $m_3(\mathrm{BEE})$, and $m_3(\mathrm{QEE})$, all change their signs at the critical point.

While the above arguments, based on the divergence of second derivative of $\omega$, guarantee the appearance of the region with negative third moments in the vicinity of the critical point, they do not tell us anything about the size of these regions in the $T$-$\mu_B$ plane. In fact, all third moments considered here become positive at sufficiently high $T$ and $\mu_B > 0$ where the system approaches a free quark and gluon system. The regions are thus limited more or less near the critical point.

The information about the behavior of the third moments at small $\mu_B$ can be extracted from the numerical results in lattice QCD. For example, with the Taylor expansion method, the thermodynamic potential is calculated to be $\omega = -c_2(T)\mu_B^2 - c_4(T)\mu_B^4 - c_6(T)\mu_B^6 - \cdots$, and one can read off the behavior of $m_3(\mathrm{BBB})$ at small $\mu_B$ as $m_3(\mathrm{BBB}) = 24[c_4(T)\mu_B + 5c_6(T)\mu_B^3 + \cdots]$. Lattice simulations indicate that $c_4(T)$ is positive definite, while $c_6(T)$ becomes negative in the high temperature phase [18]. From this result, one sees that $m_3(\mathrm{BBB})$ is positive for small $\mu_B$, while the negative $c_6(T)$ suggests that the sign of $m_3(\mathrm{BBB})$ eventually changes at sufficiently large $\mu_B$. Other moments for small $\mu_B$ can also be evaluated in the Taylor expansion method by expanding $\omega$ with respect to $T$ and $\mu_Q$. If the contour lines of vanishing third moments are close enough to the $T$ axis, the lattice simulations may be able to determine these lines. Since the region with a negative third moment should depend on the channel, combined information of signs of different third moments, and the comparison of the third moments obtained by experiments and lattice simulations, will provide a deep understanding about the state of the system in the early stage of relativistic heavy ion collisions and the QCD phase diagram.

The range of $\mu_B/T$ where lattice simulations are successfully applied, however, is limited to small $\mu_B/T$ with the present algorithms. In particular, thermodynamics around the critical point cannot be analyzed with the Taylor expansion method. In order to evaluate the qualitative behavior of the third moments in such a region, one has to resort to effective models of QCD. To make such an estimate, here we employ the two-flavor Nambu-Jona-Lasinio model [19,20] with the standard interaction $\mathcal{L}_{\mathrm{int}} = G\{(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau_i\psi)^2\}$, where $\psi$ denotes the quark field. For the model parameters, we take the values determined in Ref. [19]; $G = 5.5\ \mathrm{GeV}^{-2}$, the current quark mass $m = 5.5$ MeV, and the three-momentum cutoff $\Lambda = 631$ MeV. For the isospin symmetric matter, this model gives a first order phase transition at large $\mu_B$, as shown on the bottom surface of Fig. 1 by the bold line. The critical point is at $(T, \mu_B) \simeq (48, 980)$ MeV.

In Fig. 1, we also show the $T$ and $\mu_B$ dependence of $T\chi_B$ calculated in the mean-field approximation. One observes that $\chi_B$ diverges at the critical point, and the peak structure well survives along the crossover line up to higher temperatures [12]. The region where each moment becomes negative in the $T$-$\mu_B$ plane is shown in Fig. 2. One sees that all the moments become negative on the far side of the critical point as it should be, whereas the extent of the region depends on the channel. The figure shows that areas

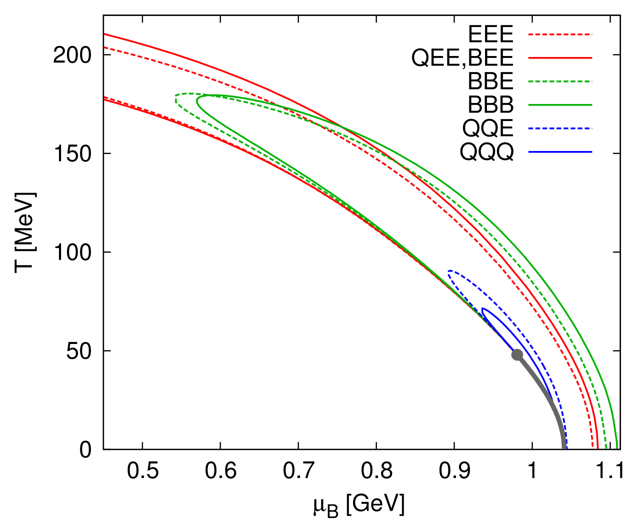


FIG. 2 (color online). Regions where third moments take negative values in the $T$-$\mu_B$ plane. The regions are inside the boundaries given by the lines.

with $m_3(\mathrm{BBB}) < 0$ and $m_3(\mathrm{BBE}) < 0$ extend to much lower $\mu_\mathrm{B}$ and much higher $T$ than the critical point. This suggests that even if the critical point is located at high $\mu_\mathrm{B}$, the negative third moments can be observed by heavy ion collision experiments. The figure also shows that the areas have considerable thicknesses along the radial direction. Since the system stays near the phase transition line considerably long regardless of the order of the phase transition, first order or crossover, once the state on the far side is created, negative third moments are very likely to be formed and observed. The wide regions of negative moments also indicate that they are hardly affected by critical slowing down and finite volume effects during the dynamical evolution of fireballs.

Figure 2 also shows that areas with negative $m_3(\mathrm{EEE})$, $m_3(\mathrm{QEE})$, and $m_3(\mathrm{BEE})$ are much larger than those of the other moments in the $T$-$\mu_\mathrm{B}$ plane; although not shown in the figure, these areas extend even to the $T$ axis. The behaviors of $m_3(\mathrm{EEE})$ and $m_3(c\mathrm{EE})$ near the $T$ axis can be checked directly by the lattice simulations. If the range of $T$ satisfying $m_3(\mathrm{EEE}) < 0$ is sufficiently wide at $\mu_\mathrm{B} = 0$, it is possible that the negative third moments are measured even at the RHIC and LHC energies. Whether the negative moments survive or not in this case depends on the diffusion time of the energy density, in other words the heat conductivity. One can thus use the signs of $m_3(\mathrm{EEE})$ and $m_3(c\mathrm{EE})$ to estimate the diffusion time of the charges and energy. The third moments $m_3(\mathrm{QQQ})$ and $m_3(\mathrm{QQE})$, on the other hand, become negative only in small regions near the critical point. These behaviors come from the large contribution of $\chi_\mathrm{I}$ in Eq. (6).

It should be, however, remembered that the results in Figs. 1 and 2, are obtained in an effective model. In particular, the model employed here gives the critical point at relatively low $T$ and high $\mu_\mathrm{B}$ [2]. If the critical point is at much lower $\mu_\mathrm{B}$, the areas with negative moments in Fig. 2 should also move toward lower $\mu_\mathrm{B}$ and higher $T$.

In this Letter, we have pointed out that the third moments of conserved charges, the net baryon and electric charge numbers and the energy, carry more information on the state around the QCD phase boundary than usual fluctuation observables. They change signs at the phase boundary corresponding to the existence of the peaks of susceptibilities. If the negative third moments grow at early stage of the time evolution of fireball created in the collisions and if the diffusion of charges is slow enough, then the negative third moments will be measured experimentally through event-by-event analyses. Once such signals are measured, they serve as direct evidence that the peak structure of corresponding susceptibility exists in the phase diagram of QCD, and that the matter on the far side of the phase transition, i.e., the quark-gluon plasma, is created. The combination of the third moments of different channels, and their comparison with the numerical results in lattice QCD, will bring various information on the phase structure and initial states created in heavy ion collisions at different energies.

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