## Hierarchical Energy Relaxation in Mesoscopic Tunnel Junctions: Effect of a Nonequilibrium Environment on Low-Temperature Transport

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We develop a theory of far from the equilibrium transport in arrays of tunnel junctions. We find that if the rate of the electron-electron interactions exceeds the rate of the electron-phonon energy exchange, the energy relaxation ensuring the charge transfer may occur sequentially. In particular, cotunneling transport in arrays of junctions is dominated by the relaxation via the intermediate bosonic *environment*, the electron-hole excitations, rather than by the electron-phonon mechanism. The current-voltage characteristics are highly sensitive to the spectrum of the environmental modes and to the applied bias, which sets the lower bound for the effective temperature. We demonstrate that the energy gap in the electron-hole spectrum which opens below some critical temperature  $T^*$  due to long-range Coulomb interactions gives rise to the suppression of the tunneling current.

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Transport in mesoscopic tunnel junctions is ensured by the energy exchange between the tunneling charge carriers and energy reservoirs: since the electronic energy levels at the banks of the mesoscopic junctions are, in general, different, the tunneling is impossible unless there is a subsystem of excitations capable of accommodation of this energy difference [1-7]. Intense studies of nanostructured and disordered systems including Josephson junctions [8], mesoscopic superconductors [9], patterned superconducting films [10], highly disordered superconducting and semiconducting films [11–14] reveal a prime importance of the out-of-equilibrium properties of an environment to which the tunneling charge carriers relax the energy. Notably, the relaxation processes can be mediated not only by phonons but by the energy exchange with the electromagnetic environment [1,3-7,15,16] and with the electron-hole (e-h) pairs generated by the tunneling carriers [17,18]. The energy relaxation in mesoscopic tunnel junctions in the case where the energy exchange between the tunneling carriers and the electromagnetic and/or electron-hole reservoir,  $1/\tau_{env-e}$ , is comparable to the rate of the energy loss to the phonon thermostat,  $1/\tau_{env\to bath}$ , was analyzed in [19]. In this Letter we develop a general approach to the description of the strongly nonequilibrium processes where  $1/\tau_{env-e} \gg 1/\tau_{env\rightarrow bath}$  and show that the energy relaxation enabling the tunneling current occurs in two stages: (i) The energy relaxation from the tunneling charges to the intermediate bosonic modes (electromagnetic or electron-hole excitations) which we hereafter call the environment; and (ii) The energy transfer from the environment to the phonon thermostat, to which we will be further referring as to a *bath*.

We demonstrate that the transport is controlled by the first stage and is thus critically sensitive to the spectrum of the environmental modes. At the same time, the passing PACS numbers: 74.50.+r, 72.10.-d, 73.23.-b, 73.63.-b

current drives the environment out of the equilibrium, and the environment spectrum and effective temperature may become bias-dependent themselves. We derive the coupled kinetics equations for the charge carriers and out-ofequilibrium bosonic environment and apply our technique to tunneling transport in large arrays of normal and superconducting junctions.

A single junction.—First, we consider a tunnel junction between two bulk metallic electrodes biased by the external voltage V; see Fig. 1(a). A general formula for the tunneling current reads

$$I = e(\vec{\Gamma} - \vec{\Gamma}), \tag{1}$$

where  $\Gamma$  ( $\Gamma$ ) is the tunneling rate from the left (right) to the right (left), and, for a single junction,

$$\vec{\Gamma} = \frac{1}{R_T} \int_{\epsilon \epsilon'} f_{\epsilon}^{(1)} (1 - f_{\epsilon'}^{(2)}) P^<(\epsilon - \epsilon'), \qquad (2)$$



FIG. 1 (color online). (a) The effective circuit for the tunnel junction subject to bias V and with the environment having the impedance Z. (b)–(c) Diagrammatic expansion of  $P^{<}$  to the first and the second orders in  $\rho$ , respectively. The solid lines represent propagation of electrons, the dashed lines denote the environment excitations. The vertex with the two electron lines and one dashed line carries a factor  $G_T \rho(\omega)/\omega$ , the "two dashed-lines vertex" corresponds to  $G_T \rho(\omega) \rho(\omega')/(\omega \omega')$ .

where  $f^{(1,2)}$  are the electronic distribution functions within the electrodes,  $P^{<}(\epsilon)$  is the probability for the charge carrier to lose the energy *E* to the environment, and  $R_T$  is the bare tunnel resistance, representing the interaction of electrons with the bath. The backward scattering rate,  $\tilde{\Gamma} \propto \int_{\epsilon\epsilon'} f_{\epsilon'}^{(2)} (1 - f_{\epsilon'}^{(1)}) P^{<}(\epsilon - \epsilon')$ . If an intermediate environment is absent and the relaxation is provided by the phonon bath, then  $P^{<}(\epsilon) = \delta(\epsilon)$  and Eq. (2) reproduces the conventional Ohm law. The quasiequilibrium situation where the distribution functions of the environmental modes  $N_{\omega}$ are Bose distributions parameterized by the equilibrium temperature was discussed in [2,5,20]. In a general, far from the equilibrium case, we find

$$P^{<}(E) = \int_{-\infty}^{\infty} dt \exp[J(t) + iEt], \qquad (3)$$

$$J(t) = 2 \int_0^\infty \frac{d\omega}{\omega} \rho(\omega) F(\omega), \qquad (4)$$

$$F(\omega) = [N_{\omega}e^{i\omega t} + (1+N_{\omega})e^{-i\omega t} - B_{\omega}].$$
 (5)

Here  $\exp[J(t)]$  is the nonequilibrium generalization of the Feynman-Vernon influence functional [21] reflecting that tunneling electrons acquire random phases due to interactions with the environment, represented by a set of oscillators with the nonequilibrium distribution of modes,  $N_{\omega}$ . The latter is defined by the kinetic equation with the scattering integral describing the energy exchange between environmental modes and tunneling electrons. Terms proportional to the  $N_{\omega}$  and  $1 + N_{\omega}$  correspond to the absorbed and emitted environmental excitations, respectively. The combination  $B_{\omega} = 1 + 2N_{\omega}$  is the kernel of the timeindependent contribution to J describing the elastic interaction of the tunneling electron with the environmental modes and having the structure of the Debye-Waller factor. In an equilibrium,  $N_{\omega}$  reduces to the Bose-function and the functional  $P^{<}$  recovers the result by Ref. [20]. The spectral probability of the electron-(electromagnetic) environment interaction is  $\rho(\omega) = \operatorname{Re}[Z_t(\omega)]/R_0$ , where  $Z_t =$ 

 $1/[iC\omega + Z(\omega)^{-1}]$  is the total circuit impedance, Z is the environment impedance, C is the junction capacitance, and  $R_0$  is the quantum resistance [5]. Proceeding analogously to Ref. [22], one finds the spectral probability corresponding to the electron-environment interaction within each electrode as  $\rho_n(\omega) = 2 \text{Im} \int_{\mathbf{q}} \tilde{U}_n / (D_n q^2 - i\omega)^2$ , n = 1, 2, and that for the interaction across the junction,  $\rho_{12}(\omega) = -2 \text{Im} \int_{\mathbf{q}} \tilde{U}_{12} / [(D_1 q^2 - i\omega)(D_2 q^2 - i\omega)],$ where  $D_{1(2)}$  are diffusion coefficients within respective electrodes, and  $U_{1(2)}$  are the dynamically screened Coulomb interactions within (across) the electrodes. The form of  $\rho_{12}(\omega)$  [and in more general case of  $\rho(\omega)$ ] depends on the structure of the environmental excitations spectrum and, thus, on the external bias. The latter is especially important in the array of highly transparent junctions where  $\rho(\omega)$  is different for elastic and inelastic processes [1,23]. In particular, for the *e*-*h* environment with constant  $\tilde{U}$ , one should cut off the (diverging) integral at  $q_T =$  $\sqrt{1/\tau_{\varphi}(T_{\rm eff})D}$  [24], when calculating  $\rho(0)$ . Here  $\tau_{\varphi}$  is the electron inelastic time and  $T_{\rm eff}$  (bias dependent) effective temperature of the environment defined below. This allows us to formulate a recipe: if in an equilibrium  $\rho = \rho(\omega, T)$ then in an out-of-equilibrium state  $\rho = \rho(\omega, T_{\text{eff}})$ .

To close the set of formulas (1)–(5) one has to add the kinetic equations (KE) for the boson distribution functions  $N_{\omega}$ . To derive these KE we use a semiphenomenological kinetic approach of [25] and express the current of Eq. (1) through the electronic distribution function as  $I = \int_{\epsilon_1} [df_{\epsilon_1}^{(1)}/dt] v_1$ . Here  $v_{1(2)}$  is the density of states in the lead 1(2) and  $df_{\epsilon_1}^{(1)}/dt = I_{col}$ , where  $I_{col}$  is the collision integral describing the evolution of the electronic distribution function due to energy and/or momentum transfer processes. Expanding further  $P^<$  with respect to  $\rho$  we obtain, in the zero order in  $N_{\omega}$ , the collision integral in a form  $I_{col}^{(0)} = -\int W_{12}[f_{\epsilon_1}^{(1)}(1-f_{\epsilon_2}^{(2)}) - f_{\epsilon_2}^{(2)}(1-f_{\epsilon_1}^{(1)})]\delta(\epsilon_1 - \epsilon_2)v_2d\epsilon_2$ , where  $W_{12} = 1/v_1v_2R_T$  is proportional to the bare probability for an electron to be transmitted from one lead to the other. In the first order

$$\frac{df_{\epsilon_1}^{(1)}}{dt} = -\int d\omega \nu_{\omega} \nu_2 d\epsilon_2 \left(\frac{\rho}{\omega \nu_{\omega}}\right) W_{12} \{\delta(\epsilon_{12} - \omega) [f_{\epsilon_1}^{(1)}(N_{\omega} + 1)(1 - f_{\epsilon_2}^{(2)}) - (1 - f_{\epsilon_1}^{(1)})N_{\omega} f_{\epsilon_2}^{(2)}] \\
+ \delta(\epsilon_{12} + \omega) [f_{\epsilon_1}^{(1)}N_{\omega}(1 - f_{\epsilon_2}^{(2)}) - (1 - f_{\epsilon_1}^{(1)})(N_{\omega} + 1)f_{\epsilon_2}^{(2)}]\},$$
(6)

where  $\epsilon_{12} = \epsilon_1 - \epsilon_2$  and  $\nu_{\omega}$  is the density of environmental states [26]. The structure of  $I_{\rm col}^{(1)}$  is identical to that of the electron-phonon scattering integral in metals [25], where  $N_{\omega}$  would stand for the phonon distribution functions. The quantity  $\rho/\omega\nu_{\omega}$  is proportional to the probability of the electron-environment scattering.

The collision integral dual to  $I_{col}^{(1)}$  and describing the evolution of  $N_{\omega}$  is derived analogously, and the resulting kinetic equation is

$$\left(\frac{dN_{\omega}}{dt}\right)_{e-\text{env}} = -\frac{A_{\rho}(\omega)}{\nu_{\omega}R_{T}} [N_{\omega}(1+n_{\omega}) - (1+N_{\omega})n_{\omega}], \quad (7)$$

where *A* is the numerical factor of order of unity,  $n_{\omega}$  is the electron-hole pairs distribution function. The scattering integral in Eq. (7) is also identical by its structure to the phonon-electron scattering integral in metals [25]. For the electron-hole environment  $(i = 1, 2 \text{ label the electrode in which the pair is located}), one has <math>n_{\omega}^{(i)} = (1/\omega) \int_{\epsilon} f_{\epsilon_+}^{(i)} (1 - f_{\epsilon_-}^{(i)})$ , where  $\epsilon_{\pm} = \epsilon \pm \omega/2$ ; this agrees with the results of Ref. [27] where the nonequilibrium boson distribution function is equivalent to our  $1 + 2n_{\omega}$ . If electrons and holes belong to different electrodes,  $n_{\omega}^{(ij)} = (2\omega)^{-1} \times \int_{\epsilon} f_{\epsilon_+}^{(i)} \sigma_{ij}^x (1 - f_{\epsilon_-}^{(j)}), \hat{\sigma}^x$  being the Pauli matrix. From (7)

one estimates the rate of the energy exchange between the environment and the tunneling electrons as:  $1/\tau_{env-e} =$  $\rho(\omega)/\nu_{\omega}R_{T}$ . Now one has to compare  $1/\tau_{env-e}$  with the rate of the interaction of the environment modes with the (phonon) bath,  $1/\tau_{env\rightarrow bath}(\omega)$ . For the electron-hole environment,  $1/\tau_{\rm env \rightarrow bath}(\omega)$  is determined from Eq. (7) to which the electron-phonon scattering integral is added. If  $\tau_{\rm env \rightarrow bath} \gg \tau_{\rm env-e},$  the two-stage relaxation takes place and the characteristic energy transfer from tunneling current is  $\omega \sim \max\{T_e, V\}$ , where  $T_e$  is the electronic temperature in the leads. The electromagnetic environment mediates the two-stage relaxation in the case where Ohmic losses occur in a LC superconducting line and are small [26]. To take a typical example, in aluminum mesoscopic samples  $\tau_{env-e} = 10^{-8}$  sec and  $\tau_{env\rightarrow bath} =$  $10^{-6}$  sec [16], so the conditions for the two-stage relaxation are realized. Then the distribution functions,  $N_{\omega}$ , deviate significantly from the Bose distribution with the temperature of the phonon bath. They should be determined from the condition that the collision integral of the environmental modes with the e-h pairs accompanying the current flow becomes zero, and Eq. (7) yields  $N_{\omega} \approx n_{\omega}^{(12)}$ . If  $T_e \ll V$ , then  $N_{\omega}$  can be approximated by the Bose function with some effective temperature  $T_{\rm eff}$  at  $\omega < V =$  $T_{\rm eff}$  and  $N_{\omega} = 0$  at  $\omega > T_{\rm eff}$  (the emission of the excitations with the energy larger than V is forbidden), and

$$T_{\rm eff} \equiv \lim_{\omega \to 0} \omega N_{\omega} = 0.5V \coth(V/2T).$$
(8)

This result shows that the system with the environment well isolated from the bath cannot be cooled below  $T_{\rm eff}$ . Note that the coth expression for  $T_{\rm eff}$  obtained in the first approximation in  $\rho$ . In general case,  $T_{\rm eff}$  depends on  $\rho$ .

Equations (1)–(7) give the full description of the kinetics of the tunneling junction in a nonequilibrium environment. To derive the *I*-V characteristics we find  $N_{\omega} \approx n_{\omega}^{(12)}$  and plug it into Eqs. (1)–(4). Introducing the parameters  $g^{-1} = \rho(0)$  and  $\Lambda$ , the characteristic frequency of the  $\rho(\omega)$  decay [for the Ohmic model [20],  $\rho = g^{-1}/\{1 + (\omega/\Lambda)^2\}$  and  $\Lambda/g$  is of the order of the charging energy of the tunnel junction], we find

$$I \sim \frac{V}{R_T} \ln \frac{\Lambda}{V},\tag{9}$$

in the interval  $T \ll V \ll \Lambda$ , where  $T_{\text{eff}} \simeq V$ . Note that I(V) given by Eq. (9) differs from the power law dependence obtained in [20] for  $T_e = T_{\text{eff}} = 0$ . This shows that tuning the environment one can control the I(V) characteristics of the tunnel junction (the gating effect). At high voltages,  $V \gg \Lambda$ , one finds

$$I(V) \simeq (V - \Delta_{\infty})/R_T, \tag{10}$$

$$\begin{split} &\Delta_{\infty} = iJ'(0) = 2\int_{0}^{\infty} d\omega \rho_{\omega} [1 + N_{\omega}^{(\text{out})} - N_{\omega}^{(\text{in})}] \simeq \Delta_{\infty}^{(0)} \ln(\Lambda / \min\{T_{e}, T_{\text{env}}\}), \text{ where } \Delta_{\infty}^{(0)} = \Delta_{\infty} [N^{(\text{out})} = N^{(\text{in})}] \sim \Lambda / g, \\ &\text{since at } V \gg \Lambda, N_{\omega}^{(\text{out})} \simeq \Lambda / \omega \gg N_{\omega}^{(\text{in})}. \end{split}$$

Arrays of tunnel junctions.—Extending Eq. (2) onto an array comprised of N junctions one finds

$$\vec{\Gamma} = \left(\prod_{i=1}^{N} \frac{R_Q}{4\pi^2 R_i}\right) S^2 \int d\boldsymbol{\epsilon} d\boldsymbol{\epsilon}' f_1(\boldsymbol{\epsilon}) [1 - f_2(\boldsymbol{\epsilon}')] P(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}'),$$
(11)

where

$$P(E) = \int_{-\infty}^{\infty} dt \exp(iEt) \left\{ \int_{0}^{\infty} d\omega \frac{\rho(\omega)}{\omega} \times \prod_{j \le N-1} [N_{\omega,j}^{(\text{in})} e^{i\omega t} + (1 + N_{\omega,j}^{(\text{out})}) e^{-i\omega t}] \right\}.$$
 (12)

Here  $S = E_c^{-(N-1)}N^N/(N-1)!$ , and  $E_c = e^2/2C$  is the Coulomb charging energy of a single junction (*C* is a single junction capacitance) and for the Cooper pair transport  $e \rightarrow 2e$ . Equations (11) and (12) were derived in a first order in tunneling Hamiltonian. Shown in Fig. 2 is a diagrammatic representation of Eq. (12) for N = 3.

A generalization of the results obtained for a single junction including the structure of the collision integral and the concept of the effective temperature Eq. (8), onto large arrays is straightforward. As long as temperatures are not extremely low [17], the charge transfer in large arrays is dominated by the inelastic cotunneling and the two-stage energy relaxation. The tunneling carriers generate e-h pairs [17,18] serving as an environment exchanging the energy with the tunneling current and then slowly losing it to the bath. It is instructive to consider a two-dimensional array of superconducting tunnel junctions. On the distances L < $\lambda = \sqrt{C/C_0}$ , where  $C_0$  is the capacitance of a single junction to the ground, the Coulomb interaction between charges is logarithmic. If the size of an array does not exceed  $\lambda$ , the *e*-*h* plasma comprising the environment experiences the charge Berezinskii-Kosterlitz-Thouless



FIG. 2 (color online). (a) The single electron two-islands' circuit. (b)–(e) Diagrams describing the forward inelastic cotunneling rate. The "up" arrows stand for the *e*-*h* pairs excited during the cotunneling and the "down" arrows correspond to the recombination of the *e*-*h* pairs. The vertices shown by boxes are proportional to the probability of an elemental *e*-*h* pair excitation,  $\rho(\omega)/\omega$ .

(BKT) transition [28,29] at  $T = T_{BKT} \simeq E_c$  [29]. This implies that at  $T \simeq E_c$  the energy gap  $T^*$  opens in the spectrum of unbound electrons and holes and, as a result,  $\rho(\omega)$  vanishes in the interval  $0 < \omega < T^*$ . One than sees from Eq. (11) that opening the gap suppresses both Cooper pairs- and normal quasiparticle currents in the superconducting tunneling array at  $T < E_c$ . Analyzing contribution from higher orders into cotunneling process, one finds that the current suppression holds in all orders. This picture applies to the films close to superconductor-insulator transition (SIT) [12]. Indeed, near the SIT the dielectric constant  $\varepsilon$  of the film diverges [30] and on the distances  $L < \varepsilon d$ , where d is the film thickness, the 2D e-h environment experiences the BKT transition. Thus opening the gap in the electron-hole spectrum due to long-range Coulomb effects and the resulting suppression of the tunneling current offers a microscopic mechanism for the insulator-to-superinsulator transition [31,32].

The notion of the two-stage relaxation is a key to resolving the controversy of the variable range hopping (VRH) conductivity in both doped semiconductors [33,34] and disordered superconducting films [12]: the observed universal preexponential factor indicates that the energy relaxation is due to electron-electron (*e-e*) rather than the electron-phonon interactions. On the other hand, according to [35,36] *e-e* relaxation cannot ensure a finite conductivity below the so called many-body localization temperature [36]. The sequential relaxation of hopping electrons via the *e-h* environment, which further transfers energy to the phonon bath implies that the prefactor in hopping conductivity is indeed proportional to  $e^2/\hbar$ .

In conclusion, we have developed a quantitative description of the highly nonequilibrium tunneling transport in arrays of tunnel junctions in the limit  $1/\tau_{env-e} \gg 1/\tau_{env\rightarrow bath}$  and demonstrated that the low-temperature relaxation ensuring the tunneling current occurs via an intermediate electromagnetic and/or *e-h* pairs environment. We argued that the onset of the gap in the spectrum of environmental excitations suppresses the tunneling current. In particular, the gap due to Coulomb interactions in superconducting arrays offers a microscopic mechanism for the insulator-superinsulator transition.

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