Spectrum of Weak Magnetohydrodynamic Turbulence

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Turbulence of magnetohydrodynamic waves in nature and in the laboratory is generally cross-helical or nonbalanced, in that the energies of Alfvén waves moving in opposite directions along the guide magnetic field are unequal. Based on high-resolution numerical simulations it is proposed that such turbulence spontaneously generates a condensate of the residual energy $E_v - E_b$ at small field-parallel wave numbers. As a result, the energy spectra of Alfvén waves are generally not scale invariant in an inertial interval of limited extent. In the limit of an infinite Reynolds number, the universality is asymptotically restored at large wave numbers, and both spectra attain the scaling $E(k) \propto k_\perp^{-2}$. The generation of a condensate is apparently related to the breakdown of mirror symmetry in nonbalanced turbulence.

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Introduction.—Magnetohydrodynamic (MHD) turbulence naturally occurs in a variety of plasmas, ranging from the interstellar medium to the solar wind to laboratory fusion devices. When the compressibility effects can be neglected, the MHD equations take an especially simple form in the so-called Elsässer variables,

$$\left(\frac{\partial}{\partial t} \mp \mathbf{v}_A \cdot \nabla\right)\mathbf{z}^{\pm} + (\mathbf{z}^{\mp} \cdot \nabla)\mathbf{z}^{\pm} = -\nabla P + \mathbf{f}^{\pm}, \quad (1)$$

where $\mathbf{z}^{\pm} = \mathbf{v} \pm \mathbf{b}$, $\mathbf{v}$ is the fluctuating plasma velocity, $\mathbf{b}$ is the fluctuating magnetic field normalized by $\sqrt{4\pi\rho_0}$, $\mathbf{v}_A = \mathbf{B}_0/\sqrt{4\pi\rho_0}$ is the Alfvén velocity associated with the uniform background magnetic field $\mathbf{B}_0$, $P = (p/\rho_0 + b^2/2)$ includes the plasma pressure p and the magnetic pressure, ρ_0 is the constant plasma density, $\mathbf{f}^{\pm}$ represents the mechanisms driving turbulence, and small dissipation due to viscosity and resistivity is neglected. In the absence of dissipation, both energies $E^+ = \langle |z^+|^2 \rangle$ and $E^- = \langle |z^-|^2 \rangle$ are conserved, which is equivalent to the conservation of total energy and cross-helicity [1].

The linear terms $(\mathbf{v}_A \cdot \nabla)\mathbf{z}^{\pm}$ describe advection of Alfvén wave packets along the guide field, while the nonlinear interaction terms, $(\mathbf{z}^{\mp} \cdot \nabla)\mathbf{z}^{\pm}$, are responsible for energy redistribution over scales. Depending on the driving force, turbulence can be either weak or strong in a certain range of scales. Denote b_λ as the rms magnetic fluctuations at the field-perpendicular scale $\lambda \propto 1/k_\perp$, and assume that the typical field-parallel wave vector of such fluctuations is $k_\|$. Then the turbulence is weak when the linear terms dominate, $k_\| v_A \gg k_\perp b_\lambda$, and it is strong otherwise.

Weak MHD turbulence may play a role in laboratory devices, in the solar wind, in the solar corona, in planetary magnetospheres, and in the interstellar medium, as indicated by energy spectra somewhat steeper that the Kolmogorov one [1–4]. For a general driving force weak MHD turbulence is typically observed at large scales in the inertial interval, while at small scales turbulence eventually becomes strong. Weak MHD turbulence admits a fuller analytical treatment compared to strong turbulence, thus providing a test bed for fundamental ideas in the theory of MHD turbulence, such as scale invariant fluxes of conserved quantities, Kolmogorov-like spectra, locality, anisotropy of turbulence; see [5,6].

When the nonlinear interaction is absent, the solution of (1) is an ensemble of shear-Alfvén and pseudo-Alfvén waves propagating along the guide field B_0 with the velocities $\pm\mathbf{v}_A$. The small nonlinear terms then can be taken into account perturbatively, and the spectrum of turbulence can be derived using the general methods of the theory of weak turbulence.

Spectra of MHD turbulence were first studied by Iroshnikov [7] and Kraichnan [8]. Those early works realized the role of the guide field in mediating the turbulent cascade; however, they assumed the small-scale fluctuations to be isotropic. Over the years, the assumption of isotropy proved to be incorrect (e.g., [1,9]). Anisotropic spectra of weak MHD turbulence were addressed by Ng and Bhattacharjee [10] and Goldreich and Sridhar [11] based on dimensional arguments, and a comprehensive analytic framework was developed by Galtier *et al.* [12]. The latter theory derives the kinetic equations for evolution of the spectral energies $e^{\pm}(\mathbf{k}) = \langle |\mathbf{z}^{\pm}(\mathbf{k})|^2 \rangle$, and has the following main results.

First, the spectral energies are transferred in the direction of large $k_\perp$, and the universal regime of weak turbulence is established at $k_\perp \gg k_\|$. In the universal regime, the dynamics are dominated by shear-Alfvén waves. We shall therefore consider only shear-Alfvén waves and keep the same notation $e^{\pm}(\mathbf{k})$ for their energies. It is also customary to use the phase-space-volume compensated spectra, $E^{\pm}(\mathbf{k}) = e^{\pm}(\mathbf{k})2\pi k_\perp$. Second, the predicted spectra are not unique, but form a one-parameter family, $E^{\pm}(k_\perp) \propto k_\perp^{-2\pm\alpha}$. The solutions with $\alpha \neq 0$ correspond to unequal fluxes of the $E^{\pm}$ energies over scales; we denote these fluxes ϵ^+ and ϵ^-. The spectral parameter α is then uniquely defined once the flux ratio ϵ^+/ϵ^- is specified. In the balanced case, $\epsilon^+ = \epsilon^-$, the energy spectrum is $E^{\pm}(k_\perp) \propto k_\perp^{-2}$ [10–13].

MHD turbulence in nature and in the laboratory is often driven by localized sources (e.g., solar wind, antennas, localized instabilities) and therefore it is generally nonbalanced. Nonbalanced turbulence possesses nonzero cross-helicity, $\int \mathbf{v}\cdot\mathbf{b}d^3x \neq 0$; therefore, it is also called "cross-helical." Nonbalanced MHD turbulence has recently attracted considerable interest [14–20].

In the present Letter we study weak nonbalanced MHD turbulence in a series of high-resolution numerical simulations. The results reveal puzzling contradictions with the theory. While in the balanced case the numerics confirm the analytic prediction $E(k_\perp) \propto k_\perp^{-2}$ (cf. [2,13]), in a general nonbalanced case simulations disagree with the theory. When we drive turbulence with unequal rates $\epsilon^\pm$, the resulting spectra $E^\pm$ turn out to be not defined by the ratio ϵ^+/ϵ^-. Rather, they depend on the Reynolds number and approach $k_\perp^{-2}$ at large $k_\perp$ as the Reynolds number increases.

To resolve this contradiction, we propose that driven weak MHD turbulence generates the residual energy condensate $\langle \mathbf{z}^+(\mathbf{k})\cdot\mathbf{z}^-(-\mathbf{k})\rangle = v^2 - b^2 \neq 0$ at $k_\parallel = 0$. This condensate has been assumed to be zero in the standard derivation [10–12]; therefore, its presence in our model requires an explanation. The Alfvén wave fluctuations obey $\mathbf{v} = \pm\mathbf{b}$, in which case the residual energy vanishes. However, at $k_\parallel = 0$ fluctuations are not waves, and the Alfvénic condition should not be necessarily satisfied. We further propose that the generation of the condensate is a consequence of the breakdown of the mirror symmetry in nonbalanced turbulence.

Kinetic equations for weak MHD turbulence.—In this section we discuss the predictions of the standard model for weak MHD turbulence developed in [10–12,17]. The kinetic equations for the evolution of the shear-Alfvén energies, derived by Galtier *et al.* [12], have the form

$$\partial_t e^\pm(\mathbf{k}) = \int M_{k,pq} e^\mp(\mathbf{q})[e^\pm(\mathbf{p}) - e^\pm(\mathbf{k})]\delta(q_\parallel)d_{k,pq}, \quad (2)$$

where we use the shorthand notation $M_{k,pq} = (\pi/v_A)\times(\mathbf{k}_\perp\times\mathbf{q}_\perp)^2(\mathbf{k}_\perp\cdot\mathbf{p}_\perp)^2/(k_\perp^2 p_\perp^2 q_\perp^2)$ and $d_{k,pq} = \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d^3pd^3q$. The derivation assumes that in the zeroth-order approximation, only the correlation functions $e^\pm(\mathbf{k})$ are nonzero. As shown in [12], the system (2) has a degeneracy: the right-hand side integrals vanish for any solutions of the form

$$e^\pm(\mathbf{k}) = g^\pm(k_\parallel)k_\perp^{-3\pm\alpha}, \quad (3)$$

with arbitrary functions $g^\pm(k_\parallel)$ that are smooth at $k_\parallel = 0$, and $-1 < \alpha < 1$. The degeneracy is removed by matching these solutions with the boundary conditions, that is, forcing and dissipation (e.g., [17]). To match with the forcing, one notes that different energy spectra correspond to different energy fluxes, $\epsilon^\pm$ supplied by the large-scale forcing, such that α is uniquely found if the ratio ϵ^+/ϵ^- is specified. One can show that the solution with the steeper spectrum corresponds to the larger energy flux [12].

The large-scale boundary conditions fix the slopes of the energy spectra, but do not fix their amplitudes. To fully remove the degeneracy, one further argues that at the dissipation scale the balance should be restored, that is, $e^+(k)$ should converge to $e^-(k)$. This "pinning" effect was first pointed out in [21], and its physics was discussed in greater detail in [12,17,18]. The pinning effect is indeed observed in our simulations presented below.

According to the above picture, if the rates of energy supply are fixed, then the *slopes* of the energy spectra $e^\pm(k)$ are fixed as well. If the dissipation scale is now changed, the *amplitudes* of the spectra should change as to maintain the specified slopes, and to make them converge at the dissipation scale. This conclusion, although consistent with Eqs. (2), seems to be at odds with the common intuition about turbulent systems, which suggests that small-scale dissipation should not significantly affect the large-scale fields subject to the same large-scale driving. This seeming contradiction motivated our interest in the problem.

Numerical method and results.—The universal properties of MHD turbulence with a strong guide field can be described by neglecting the field-parallel components of the fluctuating fields, associated with the pseudo-Alfvén mode [13,22]. By setting $\mathbf{z}_\parallel^\pm = 0$ in Eq. (1) we obtain the closed system of equations

$$\left(\frac{\partial}{\partial t} \mp \mathbf{v}_A\cdot\nabla_\parallel\right)\mathbf{z}^\pm + (\mathbf{z}^\mp\cdot\nabla_\perp)\mathbf{z}^\pm = -\nabla_\perp P + \mathbf{f}_\perp^\pm + \nu\nabla^2\mathbf{z}^\pm, \quad (4)$$

in which dissipation terms have been added, and we assume that viscosity is equal to resistivity in Alfvénic dimensionless units. These equations are known as the reduced MHD model originally developed for tokamak plasmas [23,24], and used in numerical simulations of various regimes of MHD turbulence. Depending on the spectral properties of the driving force, this system can describe either weak or strong MHD turbulence [13,22].

We employ a fully dealiased Fourier pseudospectral method to solve Eqs. (4) with a strong guide field ($v_A/v_{\rm rms} \sim 5$) in a rectangular periodic box, with field-perpendicular cross section $L_\perp^2 = (2\pi)^2$ and field-parallel box size $L_\parallel = 5L_\perp$. The choice of a rectangular box, as discussed in [13], allows for correct description of long-wavelength and low-frequency fluctuations.

The $z^\pm$ waves are driven independently by Gaussian random forces $\mathbf{f}_\perp^\pm$, with the variances $\sigma^\pm = \langle(\mathbf{f}_\perp^\pm)^2\rangle$. The imbalance is measured by the parameter $\gamma = (\sigma^+ - \sigma^-)/(\sigma^+ + \sigma^-)$. To ensure that the turbulence is weak, the forces have a broad $k_\parallel$ spectrum. They are applied in Fourier space at wave numbers $1 \le k_\perp \le 2$ and $(2\pi/L_\parallel) \le k_\parallel \le 16(2\pi/L_\parallel)$. The Fourier coefficients inside that range are independent Gaussian random numbers with the amplitudes chosen so that the resulting rms velocity fluctuations are of order unity. The individual random values are refreshed independently for each mode on average every $\tau = 0.05L_\perp/v_{\rm rms}$. We define the Reynolds number as $\mathrm{Re} = (L_\perp/2\pi)v_{\rm rms}/\nu$. A typical run covers from 50 to 100 crossing times at the largest scale.

As the force renovation time is much shorter than the inverse Alfvén frequencies of all the excited modes, the forcing supplies energies at controlled rates, $\epsilon^\pm = \frac{1}{2}\sigma^\pm$. According to the solution of (2), in this case the $+$ and $-$ energy slopes should be independent of the dissipation. The results of our numerical simulations are presented in Fig. 1. They demonstrate that the spectra are pinned at the dissipation scale. However, the amplitudes of the spectra at large scales are not sensitive to the dissipation. As a result, the spectral slopes change with the Reynolds number, as to gradually approach the balanced spectrum $E(k_\perp) \propto k_\perp^{-2}$ at large $k_\perp$. These numerical findings agree with the physical expectation that large-scale fields are determined solely by the large-scale forcing and are independent of the small-scale dissipation. They, however, contradict the standard model (2). In what follows we propose a resolution for this inconsistency.

A model for nonbalanced weak MHD turbulence.—To derive the model equations, we propose that nonbalanced MHD turbulence leads to the generation of a nonzero average,

$$\langle \mathbf{z}^+(\mathbf{k}) \cdot \mathbf{z}^-(\mathbf{k}') \rangle = \delta(\mathbf{k} + \mathbf{k}')e^0(k_\perp)\Delta(k_\parallel), \tag{5}$$

where $\Delta(k_\parallel)$ is concentrated at $k_\parallel = 0$. If the Elsasser fields $\mathbf{z}^+$ and $\mathbf{z}^-$ corresponded to Alfvén waves, such an average would be zero, since Alfvénic fluctuations satisfy $\mathbf{v} = \pm\mathbf{b}$. However, fluctuations at $k_\parallel = 0$ are not waves ($\omega = k_\parallel v_A = 0$), and the average (5) may not vanish. The presence of the condensate (5) means that the magnetic and kinetic energies are not in equipartition at $k_\parallel = 0$.

Physically, nonbalanced MHD turbulence is not mirror invariant, as it possesses nonzero cross-helicity, $\int(\mathbf{v}\cdot\mathbf{b})d^3x \neq 0$. Non-mirror-invariant turbulence can generate large-scale helical magnetic fields that are not in equipartition with the velocity field. In the presence of a uniform guide field, the magnetic helicity of fluctuations is not conserved, rather, it is generated by the integral $\int[\mathbf{z}^+ \times \mathbf{z}^-]_\parallel d^3x$, e.g., [12]. In our case of weak MHD turbulence, the magnetic field of the condensate is generated by the $k_\parallel = 0$ component of the same term $[\mathbf{z}^+ \times \mathbf{z}^-]_\parallel$. Based on this analogy, we propose that the condensate is a consequence of mirror-invariance breaking in nonbalanced MHD turbulence. Such a condensate is indeed observed in our numerics, see Fig. 2.

The dynamics of the condensate are not described by the weak turbulence theory, but require additional assumptions. It is worth pointing out that the same limitation holds for the original system (2). As is seen from (2), only the $e^\mp(q_\parallel = 0)$ modes are responsible for the energy transfer. However, if one applies Eq. (2) to these modes themselves, one encounters an inconsistency. The weak turbulence approximation is valid when the inverse time of nonlinear interaction is much smaller than the wave frequency. The nonlinear interaction described by the right-hand side of (2) does not vanish for $k_\parallel = 0$, while the linear frequency of the corresponding Alfvén waves, $\omega = k_\parallel v_A$, vanishes. Therefore, as noted in [12], an additional assumption of smoothness of the functions $g^\pm(k_\parallel)$ at $k_\parallel = 0$ was essential for deriving the spectra (3).

We postpone the discussion of self-consistent condensate equations for future communications. Here we demonstrate the effect that the presence of condensate provides on the turbulence spectra. As we have argued, the condensate is expected to alter the cascade dynamics in the nonbalanced case. Consider the case where the imbalance is weak, that is $\gamma \ll 1$. In this case we expect that the condensate is weak as well. We derive the equations for the energies $e^\pm(\mathbf{k})$, by proceeding along the lines of weak turbulence derivation: we expand the MHD Eqs. (1) up to the second order in the nonlinear terms, and split the forth-order correlators into the second-order ones according to the Gaussian rule. To the first order in $e^0(k_\perp)$, the resulting equations have the form

$$\begin{aligned}\partial_t e^\pm(\mathbf{k}) = &\int M_{k,pq} e^\mp(\mathbf{q})[e^\pm(\mathbf{p}) - e^\pm(\mathbf{k})]\delta(q_\parallel)d_{k,pq} \\ &+ \tilde{\Delta}(k_\parallel)\int R_{k,pq}[e^\pm(k_\perp)e^0(q_\perp) \\ &+ e^\pm(q_\perp)e^0(k_\perp)]d^\perp_{k,pq},\end{aligned} \tag{6}$$

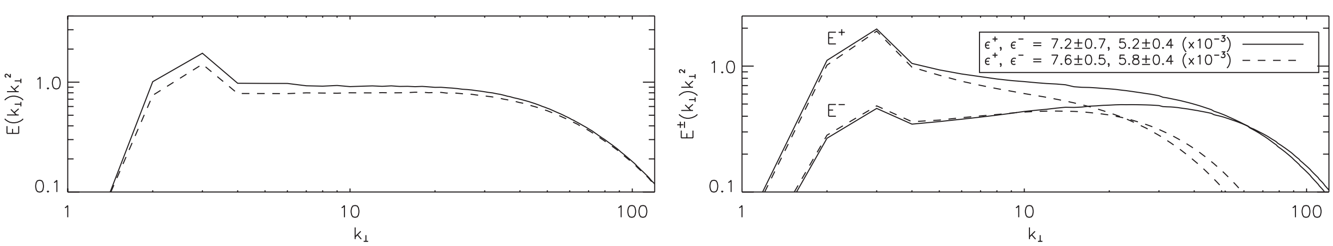

FIG. 1. Left panel: The spectra of balanced weak MHD turbulence, $k_\perp$ is measured in units of $2\pi/L_\perp$. The solid line is $E^+(k_\perp)$, the dashed line is $E^-(k_\perp)$; $\mathrm{Re} = 6000$, resolution $1024^2 \times 256$ points. Right panel: The spectra of nonbalanced weak MHD turbulence, with the imbalance parameter $\gamma = 0.17$. The solid lines denote $E^+(k_\perp)$ and $E^-(k_\perp)$, $\mathrm{Re} = 4500$, resolution $1024^2 \times 256$ points. The dashed lines show the same fields for $\mathrm{Re} = 2000$ and resolution $512^2 \times 256$ points. The inset shows the corresponding energy dissipation rates.

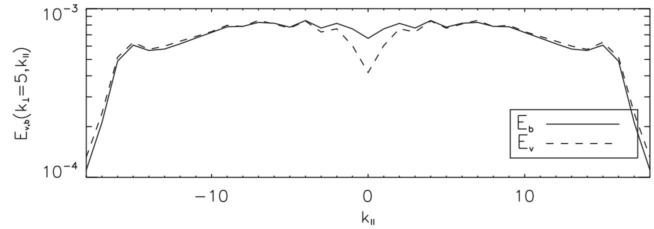

FIG. 2. The field-parallel spectra of the magnetic energy (solid line) and the kinetic energy (dashed line) at the field-perpendicular wave number $k_\perp = 5$; Re $= 2500$, resolution $512^2 \times 256$ points, $k_\parallel$ is in units of $2\pi/L_\parallel$.

where $\tilde{\Delta}(k_\parallel) = \mathrm{Re}\Delta(k_\parallel)$, $R_{k,pq} = (\pi/v_A)(\mathbf{k}_\perp \times \mathbf{q}_\perp)^2 \times (\mathbf{k}_\perp \cdot \mathbf{p}_\perp)(\mathbf{k}_\perp \cdot \mathbf{q}_\perp)/(k_\perp^2 p_\perp^2 q_\perp^2)$, and $d^\perp_{k,pq} = \delta(\mathbf{k}_\perp - \mathbf{p}_\perp - \mathbf{q}_\perp)d^2p_\perp d^2q_\perp$. The first term in (6) coincides with Eq. (2), while the second term describes the interaction with the condensate. It can be directly verified that each of the integrals in (6) conserves the Elsasser energies $E^\pm = \int e^\pm(\mathbf{k})d^3k$.

For the stationary solution, each of the integrals in (6) should vanish independently. Equating the first integral to zero does not allow one to find the spectra uniquely, rather, it leads to the one-parameter family of solutions (3). Consider the second integral that describes the interaction of $e^\pm$ with the condensate. By employing the standard methods of the weak turbulence theory, one can demonstrate that the power-law solution nullifying the first part of the second integral is unique, $e^0(k_\perp) \propto k_\perp^{-3}$. Analogously, the second part of the second integral is zero if $e^\pm(k_\perp) \propto k_\perp^{-3}$. We conclude that the presence of the condensate lifts the degeneracy of the solutions: the only possible stationary power-law spectra of weak MHD turbulence are $e^\pm(k_\perp) \propto k_\perp^{-3}$. Although we do not have the equation for the condensate, the above result allows us to predict that in order to preserve the scale invariance the condensate should have the scaling $e^0(k_\perp) \propto k_\perp^{-3}$.

Conclusions.—Based on analytic consideration and numerical simulations, we propose that weak MHD turbulence spontaneously generates a condensate of the residual energy $E_v - E_b$ at small $k_\parallel$. We argue that the condensate is a consequence of mirror-symmetry breakdown in nonbalanced turbulence. When the turbulence is balanced, the energy spectra are $E^\pm(k_\perp) \propto k_\perp^{-2}$, in agreement with the analytic prediction of [10–12]. In the balanced case the evolution of $E^\pm$ fields is not affected by the condensate. In the nonbalanced case the interaction with the condensate becomes essential, and we propose that no universal power-law spectra exist in an inertial interval of limited extent. Both spectra $E^\pm(k_\perp)$ have the large-scale amplitudes fully specified by the external forcing, and they converge at the dissipation scale. As the dissipation scale decreases, the spectral scalings (but not necessary amplitudes) approach each other at large $k_\perp$. As a result, the universal scaling $k_\perp^{-2}$ is recovered for both spectra $E^\pm(k_\perp)$ asymptotically at $k_\perp \to \infty$.

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