Fractional Topological Insulators

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We analyze generalizations of two-dimensional topological insulators which can be realized in interacting, time reversal invariant electron systems. These states, which we call fractional topological insulators, contain excitations with fractional charge and statistics in addition to protected edge modes. In the case of s^z conserving toy models, we show that a system is a fractional topological insulator if and only if σ_{sH}/e^* is odd, where σ_{sH} is the spin-Hall conductance in units of $e/2\pi$, and e^* is the elementary charge in units of e.

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Introduction.—Recently, it was realized that in two dimensions there are two distinct universality classes of time reversal invariant band insulators—topological insulators and trivial insulators [1,2]. The two kinds of insulators can be distinguished by the fact that the edge of a topological insulator contains a protected pair of gapless edge modes of opposite chiralities, while no such protected edge modes exist for a trivial insulator. Indeed, such a protected edge mode has been observed in HgTe quantum wells [3].

In interacting electron systems, many other gapped, charge-conserving electronic ground states are possible in addition to band insulators. It is natural to wonder if analogues of topological and trivial insulators exist for these more general states. That is, do some of these states have time reversal protected edge modes, while some do not? This is the issue we investigate in this Letter.

Our starting point is the standard toy model construction of topological insulators [1,2]. In this construction, one imagines a two-dimensional system of electrons in the continuum where the electrons experience a spin-dependent magnetic field $B_0\hat{z}\sigma_z$. (In a more realistic context, this kind of physics can originate from spin-orbit coupling) [1,2]. One then assumes that the electrons are noninteracting and that the electron density is tuned to an integer Landau filling $\nu = k$. The ground state is thus made up of two decoupled spin species which form integer quantum Hall states with opposite chiralities. One can show that when k is odd, this state is a topological insulator, when k is even it is a trivial insulator.

A simple way to generalize this construction is to imagine that the two spin species each form *fractional* quantum Hall (FQH) states. Such states can be realized in a toy model similar to the one described above. The only new element is that one introduces a toy electron-electron interaction where electrons of the same spin interact via a short-range two-body repulsive force, and electrons of different spin do not interact at all. This system can then be mapped onto two decoupled FQH systems of the same filling factor and opposite magnetic field [4]. Depending on the electron density and the details of the electron interaction, one can engineer scenarios where each spin species forms arbitrary FQH states.

In this Letter, we study the properties of these states and their stability to perturbations. We leave the analysis of their experimental feasibility to future work. Such states were considered by Bernevig *et al.* in the context of the fractional quantum spin-Hall effect [2]. Here, we show that these states are interacting analogues of topological or trivial insulators: some of these states, which we dub "fractional topological insulators", have time reversal protected edge modes and some, which we dub "fractional trivial insulators", do not. We find that a state is a fractional topological insulator if and only if the integer parameter σ_{sH}/e^* is odd, where σ_{sH} is the spin-Hall conductance measured in units of $e/2\pi$, and e^* is the elementary charge (e.g. smallest charge of any quasiparticle excitation), measured in units of e. This result is quite general and holds for any s^z conserving model.

We also show that every fractional trivial insulator with $1/e^*$ odd has a "partner" fractional topological insulator which is adiabatically equivalent to it in the absence of time reversal symmetry. On the other hand, when $1/e^*$ is even, we find that fractional topological insulators are not possible at all (at least for the models described above). In the course of analyzing this case, we derive a result about general electronic FQH states which may be useful in its own right: we show that if $1/e^*$ is even then σ_{xy}/e^* must also be even. One implication is that if $\sigma_{xy} = 1/2$ then e^* is at most $1/4$. Note that there are no such restrictions on the charge of the *lowest energy* excitation; in principle this can be any multiple of e^*.

An interesting example where our results are applicable occurs in the case where the two spin species each form $\nu = 1/2$ FQH states. More specifically, consider the six possibilities corresponding to the Pfaffian state [5], strong-pairing state [6], 331 state [6,7], and their particle-hole conjugates. These 6 states all have different edge structures—some states have one chiral boson mode (strong-pairing), some have two (331 and anti-strong-pairing), some have three (anti-331), and some have Majorana modes (Pfaffian and anti-Pfaffian) [8]. One might guess that some of these states are fractional topological insulators—for example, those with an odd number of chiral boson edge modes. However, this is *not* the case; these states all have $\sigma_{\rm sH} = 1/2$, $e^* = 1/4$, so that $\sigma_{\rm sH}/e^*$ is even. In fact, this is true for any $\nu = 1/2$ state.

Flux insertion argument.—We derive the above criterion using a generalization of the flux insertion argument of Fu *et al.* [9] We first review this argument in the case of the above noninteracting toy model. Consider a cylindrical geometry, and assume that the number of electrons is even. In addition, assume that the ground state is time reversal invariant when there is zero flux through the cylinder. Under these assumptions, Fu *et al.* argued that there must be at least one low-lying excited state if k is odd—even in the presence of an arbitrary time reversal invariant perturbation. To see this, start with the ground state of the toy model at zero flux and imagine adiabatically inserting half of a flux quantum $\Phi_0/2$ through the cylinder. Let us call the resulting state Ψ_1. Similarly, let Ψ_2 be the state obtained by adiabatically inserting $-\Phi_0/2$ flux through the cylinder. The state Ψ_2 can be transformed into Ψ_1 by inserting a full flux quantum—an operation which transfers k electrons with spin-up from the left edge to the right edge, and k electrons with spin-down from the right edge to the left edge. In particular, this means that the two states are orthogonal. On the other hand, Ψ_1, Ψ_2 have the same energy since they are time-reversed partners: $\Psi_2 = \mathcal{T}\Psi_1$. Thus, the system with half of a flux quantum has a degenerate, low-lying eigenstate. The key point is that this degeneracy is robust against arbitrary time reversal invariant perturbations if k is odd. The reason is that when k is odd, there are an odd number of unpaired electrons localized near each of the two edges of Ψ_1, Ψ_2. Thus, as long as the edges are well separated, Kramer's theorem guarantees that Ψ_1, Ψ_2 are degenerate (and in fact there must be two other degenerate states, in addition). The claim now follows: the robust degeneracy at half of a flux quantum implies that there is also a robust low-lying excited state at zero flux (since the insertion of half of a flux quantum cannot close a gap).

We now generalize this argument to the interacting toy models described above (or more generally, any s^z and charge-conserving system). The crucial difference with the noninteracting case is that in the more general case the ground state may have topological order. The presence of nontrivial topological order means that in a cylindrical geometry there are always at least a finite number of low-lying states—even if the edge is gapped [10]. These low-lying states belong to different topological sectors and can be distinguished from one another by measuring the Berry phase associated with moving a quasiparticle around the cylinder. Thus, to show that the edge is gapless in the general case, it is not enough to just establish that there are low-lying states at zero flux; we have to show that there are low-lying states in the same topological sector as the ground state.

Because we need to establish this stronger claim, the generalized flux insertion argument begins by inserting not $\pm\Phi_0/2$ flux but $\pm N\Phi_0/2$ flux where N is the smallest integer such that the resulting states Ψ_1, Ψ_2 are in the same topological sector. The argument then proceeds as before. One notes that Ψ_1 can be obtained from Ψ_2 by transferring $N\sigma_{\rm sH}$ spin-up electrons from the left edge to the right edge and $N\sigma_{\rm sH}$ spin-down electrons from the right edge to the left edge. If $N\sigma_{\rm sH}$ is odd, then there is a Kramers degeneracy associated with each edge. In this case, the degeneracy between Ψ_1, Ψ_2 (and the two other states) is stable and cannot be split by any time reversal invariant perturbation. Since Ψ_1, Ψ_2 are in the same topological sector, we conclude that there must be at least one low-lying state at zero flux in the same sector as the ground state. Hence the edge cannot be gapped out by any time reversal invariant perturbation.

To complete the argument, we need to determine the integer N. To this end, note that N can be equivalently defined as the minimal number of flux quanta that need to be inserted to go from an initial state to a final state in the same topological sector. It is easy to see that adiabatically inserting N flux quanta changes the Berry phase associated with braiding a quasiparticle around the cylinder by

$$\Delta\theta = 2\pi N q, \tag{1}$$

where q is the charge of the quasiparticle (in units of e). In order for the initial and final states to be in the same topological sector, $\Delta\theta$ must be a multiple of 2π for every quasiparticle. Thus, the minimal value of N is $1/e^*$, with e^* the charge of the smallest charged quasiparticle. Using this value of N in the above argument, we derive the above criterion that there is a protected edge mode whenever $\sigma_{\rm sH}/e^*$ is odd.

Microscopic analysis.—While the flux insertion argument proves that there is a protected edge mode whenever $\sigma_{\rm sH}/e^*$ is odd, it does not prove that the edge modes can be gapped out when $\sigma_{\rm sH}/e^*$ is even. In order to fill in this gap, and also to show that the two degenerate states from the argument are part of a gapless mode, we now rederive the criterion using a microscopic approach. We focus on the Abelian case for simplicity. In general, the edge of an Abelian FQH state is described by a Lagrangian density [11]

$$L_c(\phi; K, V, t) = \frac{1}{4\pi}(K^{ij}\partial_x\phi_i\partial_t\phi_j - V^{ij}\partial_x\phi_i\partial_x\phi_j + \epsilon^{\mu\nu}t^i\partial_\mu\phi_i A_\nu). \tag{2}$$

Here, ϕ is an N-component vector of fields, K is the $N\times N$ K matrix, V is the velocity matrix, t is the charge vector, and A_μ is the external vector potential. We use a normalization where electron creation operators are of the form $e^{i\theta(l)}$ with $\theta(l) \equiv l^T K\phi$ and l an integer valued N dimensional vector satisfying $l^T t = 1$.

For the time reversal symmetric systems that we study, there are N fields $\phi_i^{\uparrow}$, $\phi_i^{\downarrow}$ for each spin direction and the Lagrangian density is of the form

$$L = L_c(\phi^{\uparrow}; K, V, t) + L_c(\phi^{\downarrow}; -K, V, t). \tag{3}$$

The Lagrangian (3) has $2N$ gapless edge modes, N for each chirality. Our goal is to find the conditions under which these modes can be gapped out by charge-conserving, time reversal symmetric perturbations. The question we ask is a question of principle, and therefore we will not discuss whether particular terms exist in realistic conditions. We also will not impose a requirement of momentum or spin conservation on the terms we study, since both momentum and spin may be exchanged with impurities or an underlying lattice.

The creation operators for electrons of the two spin directions are $e^{i\theta^{\uparrow}(l)}$ and $e^{-i\theta^{\downarrow}(l)}$ with $\theta^{\uparrow}(l)$, $\theta^{\downarrow}(l)$ defined as above. We use the convention that the fields transform under time reversal as $\phi^{\uparrow} \to \phi^{\downarrow}$, $\phi^{\downarrow} \to \phi^{\uparrow} - \pi K^{-1} t$. This guarantees that electron creation operators transform as $\psi_{\uparrow}^{\dagger} \to \psi_{\downarrow}^{\dagger}$, $\psi_{\downarrow}^{\dagger} \to -\psi_{\uparrow}^{\dagger}$.

It is convenient for us to define a $2N$-dimensional vector

$$\Phi = \begin{pmatrix} \phi^{\uparrow} \\ \phi^{\downarrow} \end{pmatrix},$$

a $2N \times 2N$ K matrix

$$\mathcal{K} = \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix}$$

and a charge vector

$$\tau = \begin{pmatrix} t \\ t \end{pmatrix}.$$

Also, let

$$\Sigma_x = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

and

$$\Sigma_z = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix},$$

where $\mathbf{1}$ is the N-dimensional unit matrix. In this notation, a generic charge-conserving scattering term is of the form $U(x)e^{i\Theta(\Lambda)} + \text{H.c.}$, where $\Theta(\Lambda) \equiv \Lambda^T \mathcal{K} \Phi$, and Λ is a $2N$-dimensional integer valued vector satisfying $\Lambda^T \tau = 0$. The field Θ transforms under time reversal as

$$\mathcal{T}\, \Theta(\Lambda)\mathcal{T}^{-1} = \Theta(-\Sigma_x \Lambda) - Q(\Lambda)\pi, \tag{4}$$

where $Q(\Lambda) \equiv \frac{1}{2}\Lambda^T \Sigma_z \tau$ is the number of spins flipped by Λ. Thus, one can construct scattering terms that are even/odd under time reversal by taking

$$U_{\pm} = U(x)[\cos(\Theta(\Lambda) - \alpha(x)) \pm (-1)^Q \cos(\Theta(\Sigma_x \Lambda) + \alpha(x))]. \tag{5}$$

Single particle terms ($Q = 1$) can arise from either a Zeeman interaction with a magnetic field in the xy plane, or a spin-orbit interaction. A Zeeman interaction—which is odd under time reversal—generates terms of the form $U_Z(x)\cos(\Theta(\Lambda_Z) + \alpha(x))$, with $Q(\Lambda_Z) = 1$ and $\Sigma_x \Lambda_Z = -\Lambda_Z$. For large U_Z with appropriate spatial dependence, this term introduces a mass term to Θ. Including all the possibilities for Θ, such an interaction is sufficient to gap all the edge modes. On the other hand, a spin-orbit coupling—which does not break time reversal symmetry—generates two other kinds of terms. The first type scatters an electron from an edge mode to its time-reversed partner. It is of the form $U_{\text{so}}(x)\partial_t \Theta \cos(\Theta + \alpha(x))$. This term cannot gap any modes as it is a complete time derivative. The second type is of the form U_+ of (5), again with $Q = 1$. Such terms can gap out modes (but not all of them, if the system is a topological insulator). Terms that flip two spins can arise from time reversal symmetric electron-electron interactions. Although not easily realized, similar terms flipping more than two spins can also occur in principle.

We now examine whether a time reversal symmetric perturbation of the type (5) can gap the spectrum without spontaneously breaking time reversal symmetry. We focus on the cases $N = 1, 2$—the analysis for larger N is similar. In the single edge mode case, $N = 1$, we have $\sigma_{\text{sH}}/e^* = 1$ for all K. Thus, according to the flux insertion argument, no gap can be opened without breaking time reversal symmetry in any of these states. To see this microscopically, note that the only charge-conserving vectors are of the form $\Lambda = (n, -n)$. The corresponding perturbation is even under time reversal for even n and odd for odd n. Thus, perturbations of the form U_+ of (5) require even n, say $n = 2$. For large U, such a perturbation can open a gap in the spectrum. However, hand in hand with that it also spontaneously breaks time reversal symmetry: when U is large, the operator $\cos[\frac{e}{e^*}(\phi^{\uparrow}(x) + \phi^{\downarrow}(x))]$, which corresponds to $n = 1$ and is odd under time reversal, acquires a nonzero expectation value. Hence, it is impossible to gap the two edge modes without breaking time reversal symmetry, explicitly or spontaneously.

The case $N = 2$ is a bit more complicated. In this case, the two pairs of counter-propagating edge modes can be gapped if one can find two linearly independent charge-conserving four-component integer vectors Λ_1, Λ_2 such that (a) $\Lambda_1 = -\Sigma_x \Lambda_2$ and (b) $\Lambda_1^T \mathcal{K} \Lambda_1 = \Lambda_2^T \mathcal{K} \Lambda_2 = \Lambda_1^T \mathcal{K} \Lambda_2 = 0$. Here, the second condition is simply Haldane's criterion for gapping out FQH edge modes [12]. It guarantees that one can make a linear change of variables from Φ to Φ' such that the action for Φ' will be that of two decoupled nonchiral Luttinger liquids. The two terms in (5) will then gap the spectrum of these two liquids by freezing the values of $\Theta(\Lambda_1)$ and $\Theta(\Lambda_2)$.

In some cases, this freezing of these values can lead to a spontaneous breaking of time reversal symmetry. As in the $N = 1$ case, this can happen if the perturbation fixes the value of $\Theta(\Lambda)$ where Λ is nonprimitive—e.g., a multiple of an integer valued vector. Then an operator of the form U_- of (5) may acquire an expectation value.

We now turn to search for Λ_1 and Λ_2. It is convenient to work in a basis where all four components of the charge vector are 1. We can parameterize the matrix K as

$$K = \begin{pmatrix} b + us & b \\ b & b + vs \end{pmatrix} \tag{6}$$

with b, u, v, s integers and u and v having no common factor. In terms of these parameters, the spin-Hall conductance is $(u+v)/[(u+v)b + uvs]$ and the elementary charge is $1/[(u+v)b + uvs]$. Their ratio is then $(u + v)$, so according to the flux insertion argument the parity of $u + v$ determines whether the spectrum can be gapped.

When $u + v$ is odd, it is indeed impossible to find Λ_1, Λ_2 that satisfy (a), (b) and do not spontaneously break time reversal symmetry. Imagine one had such a solution and define $\Lambda_\pm = \Lambda_1 \pm \Lambda_2$. Then, $\Sigma_x \Lambda_- = \Lambda_-$ and $Q(\Lambda_-) = 0$ so Λ_-^T must be an integer multiple of $(1, -1, 1, -1)$. Also, $\Sigma_x \Lambda_+ = -\Lambda_+$ and $\Lambda_-^T \mathcal{K} \Lambda_+ = 0$ so Λ_+^T must be an integer multiple of $(v, u, -v, -u)$. But $\cos(\Theta(v, u, -v, -u))$ is odd under time reversal. This means that the scattering term corresponding to Λ_1, Λ_2 will spontaneously break time reversal symmetry.

On the other hand, when $u + v$ is even (so that both u, v are odd), this analysis suggests an obvious solution (Λ_1, Λ_2). We can take $\Lambda_-^T = (1, -1, 1, -1)$, $\Lambda_+^T = (v, u, -v, -u)$, so that

$$\Lambda_1^T = \tfrac{1}{2}(1 + v, -1 + u, 1 - v, -1 - u) \tag{7}$$

and $\Lambda_2 = -\Sigma_x \Lambda_1$. Note that the scattering terms corresponding to Λ_1, Λ_2 flip the spins of $(v + u)/2$ electrons so at second order they flip $(u + v)$ electrons—precisely the number needed to connect the two degenerate states discussed in the flux insertion argument.

Partner states.—Now that we have derived the σ_{sH}/e^* criterion using two different approaches, we return to ask whether the imposition of time reversal symmetry generally causes each interacting universality class to split into two (or more) subclasses, as it does in the noninteracting case. Equivalently, does each fractional topological insulator have a partner fractional trivial insulator (and vice versa) which can be adiabatically connected to it in the absence of time reversal symmetry?

The answer to this question depends on the parity of $1/e^*$ (at least for the models analyzed here). When $1/e^*$ is odd, a partner state Ψ' may be constructed for an arbitrary s^z conserving fractional topological or trivial insulator Ψ. Let Ψ' be a decoupled bilayer state, where one layer is the original state Ψ, and the other layer is a noninteracting, s^z conserving, topological insulator with spin-up or spin-down electrons at $\nu = \pm k$ (k odd). Clearly the spin-Hall conductance of Ψ' is given by $\sigma'_{\mathrm{sH}} = \sigma_{\mathrm{sH}} + k$ where σ_{sH} is the spin-Hall conductance of Ψ. Also, the elementary charge is the same as Ψ: $e'^* = e^*$. Combining these two relations, we see that $\sigma'_{\mathrm{sH}}/e'^* = \sigma_{\mathrm{sH}}/e^* + k/e^*$. Since k, $1/e^*$ are odd, $\sigma'_{\mathrm{sH}}/e'^*$ has the opposite parity from σ_{sH}/e^*. Hence Ψ' has a protected edge mode if and only if Ψ does not. On the other hand, it is clear that Ψ, Ψ' are adiabatically equivalent in the absence of time reversal symmetry since this is true for noninteracting topological or trivial insulators. We conclude that Ψ' is indeed a partner state to Ψ. Thus, the universality classes with $1/e^*$ odd split into (at least) two subclasses when time reversal symmetry is imposed.

In contrast, we now show that for an even $1/e^*$ there are no fractional topological insulators (at least for the models analyzed here), since σ_{sH}/e^* is necessarily even. For simplicity we prove the analogous statement for quantum Hall states, e.g. $1/e^*$ even implies σ_{xy}/e^* even. Imagine adiabatically inserting $N = 1/e^*$ flux quanta at a point z_0. As a consequence, a quasiparticle excitation will be created at z_0. We can compute the statistical angle for these excitations in two ways. The first way is to explicitly compute the Berry phase associated with exchanging two excitations. Using (1) with $N = 1/e^*$ and $q = \sigma_{xy}/e^*$ and dividing the result by 2 since we are interested in an exchange rather than a 2π braiding, the Berry phase is

$$\theta = \pi \frac{1}{e^*} \frac{\sigma_{xy}}{e^*}. \tag{8}$$

Note that the coefficient of π is a product of an integer σ_{xy}/e^* and an even integer $1/e^*$. The Berry phase is thus a multiple of 2π—implying that the particles are bosons. On the other hand, this quasiparticle excitation is made up of σ_{xy}/e^* electrons. We conclude that σ_{xy}/e^* is even.

Summary of results.—In this Letter, we have shown that an s^z conserving model is a fractional topological insulator if and only if σ_{sH}/e^* is odd, where σ_{sH} is the spin-Hall conductance (in units of $e/2\pi$) and e^* is the elementary charge (in units of e).

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