Hydrodynamics with Triangle Anomalies

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We consider the hydrodynamic regime of theories with quantum anomalies for global currents. We show that a hitherto discarded term in the conserved current is not only allowed by symmetries, but is in fact required by triangle anomalies and the second law of thermodynamics. This term leads to a number of new effects, one of which is chiral separation in a rotating fluid at nonzero chemical potential. The new kinetic coefficients can be expressed, in a unique fashion, through the anomaly coefficients and the equation of state. We briefly discuss the relevance of this new hydrodynamic term for physical situations, including heavy-ion collisions.

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Introduction.—Relativistic hydrodynamics is important for many questions in nuclear physics, astrophysics, and cosmology. For example, hydrodynamic models are used extensively for describing the evolution of the fireball created in heavy-ion collisions. The relativistic hydrodynamic equations were proposed many years ago [1,2]; such equations describe the dynamics of an interacting relativistic theory at large distance and time scales. The hydrodynamic variables are the local velocity $u^\mu(x)$ (satisfying $u^2 = -1$), the local temperature $T(x)$ and chemical potential(s) $\mu^a(x)$, where the index a numerates the conserved charges. The hydrodynamic equations govern the time evolution of these variables; they have the form of the conservation laws $\partial_\mu T^{\mu\nu} = 0$, $\partial_\mu j^{a\mu} = 0$, supplemented by the constitutive equations which express $T^{\mu\nu}$ and $j^{a\mu}$ in terms of u^μ, T, and μ^a. These equations are the relativistic generalization of the Navier-Stokes equations.

One feature of relativistic quantum field theory that does not have a direct counterpart in nonrelativistic physics is the presence of triangle anomalies [3,4]. For currents associated with global symmetries, the anomalies do not destroy current conservations, but are reflected in the three-point functions of the currents. When the theory is put in external background gauge fields coupled to the currents, some of the currents will no longer be conserved.

In this Letter, we show that the presence of quantum triangle anomalies leads to an important modification of the hydrodynamic equations. In other words, in a hot and dense medium quantum anomalies are expressed macroscopically. This modification should be important in many physical situations, including the quark gluon plasma where the small masses of the u and d quarks can be neglected.

In the simplest case when there is one U(1) current with a $\mathrm{U}(1)^3$ anomaly, the constitutive equation for the conserved current j^μ must contain an additional term proportional to the vorticity [5]

$$j^\mu = nu^\mu - \sigma T(g^{\mu\nu} + u^\mu u^\nu)\partial_\nu\left(\frac{\mu}{T}\right) + \xi\omega^\mu, \quad (1)$$

$$\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}u_\nu\partial_\lambda u_\rho, \quad (2)$$

where n is the charge density, σ is the conductivity, and ξ is the new kinetic coefficient.

Even in a parity-invariant theory, the vorticity-induced current $\xi\omega^\mu$ is allowed by symmetries if, e.g., j^μ is a chiral current. This term contains only one spatial derivative, and its effect is as important as those of viscosity or diffusion. Before very recently, this term had been completely overlooked. In fact, if one follows the standard textbook derivation [2], the new term seems to be disallowed by the existence of an entropy current with manifestly positive divergence, required by the second law of thermodynamics.

Recently, however, calculations using the techniques of gauge/gravity duality [6–8] within a particular model ($\mathcal{N} = 4$ super-Yang-Mills plasma with an R-charge density) give a nonzero value for ξ [9–11]. This indicates that the problem with the entropy current must be circumvented in some way.

In this Letter we show that this new term is not only allowed, but is required by anomalies. Moreover, the parity-odd kinetic coefficient ξ is completely determined by the anomaly coefficient C, defined through the divergence of the gauge-invariant current, $\partial_\mu j^\mu = -\frac{1}{8}C\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$, and the equation of state,

$$\xi = C\left(\mu^2 - \frac{2}{3}\frac{\mu^3 n}{\epsilon + P}\right), \quad (3)$$

where ϵ and P are the energy density and pressure. In the case of multiple U(1) conserved currents, the formulas are modified only slightly. Namely, Eq. (3) becomes

$$\xi^a = C^{abc}\mu^b\mu^c - \frac{2}{3}n^a C^{bcd}\frac{\mu^b\mu^c\mu^d}{\epsilon + P}, \tag{4}$$

where a, b, c numerate the currents, C^{abc} is symmetric under permutations of indices and is determined from the anomalies, $\partial_\mu j^{a\mu} = -\frac{1}{8}C^{abc}\epsilon^{\mu\nu\alpha\beta}F^b_{\mu\nu}F^c_{\alpha\beta}$. Equations (3) and (4) are the central results of this Letter.

The physical meaning of these new terms can be made explicit by the following example. Consider a volume of rotating quark matter, made of massless u and d quarks, at baryon chemical potential μ. For a moment let us neglect instanton effects, so the $\mathrm{U}(1)_A$ current $j_5^\mu = \bar{q}\gamma^\mu\gamma^5 q$ is conserved. Because of the triangle anomaly in the three-point correlators of j_5^μ with two baryon currents, Eq. (4) implies that axial current will flow along the axis of rotation: $\langle j_5^\mu\rangle = \frac{3}{\pi^2}\omega^\mu$. This can be thought of as chiral separation: left- and right-handed quarks move with slightly different average momentum, creating an axial current. Since this effect owes its existence to triangle anomalies, we do not expect it to be present in nonrelativistic normal fluids. Instead, in nonrelativistic fluids, chiral separation appears in higher orders in derivative expansion [12]. Analogously, in the presence of baryon and isospin chemical potential, the axial isospin current $\bar{q}\gamma^\mu\gamma^5\tau^3 q$ flows along the rotation axis.

Entropy current in hydrodynamics with anomalies.—For simplicity consider first a relativistic fluid with one conserved charge, with a $\mathrm{U}(1)^3$ anomaly. To constrain the hydrodynamic equation, we turn on a slowly-varying background gauge field A_μ coupled to the current j_μ. We take the strength of A_μ to be of the same order as the temperature and the chemical potential, so $A_\mu \sim O(p^0)$ and $F_{\mu\nu} \sim O(p)$. As in first-order hydrodynamics, we keep terms of order $O(p)$ in the constitutive equations for $T^{\mu\nu}$ and j^μ (or terms of order $O(p^2)$ in the equations of motion). Note that A_μ is not dynamical.

In the presence of an external background field the hydrodynamic equations obtain the form

$$\partial_\mu T^{\mu\nu} = F^{\nu\lambda}j_\lambda, \qquad \partial_\mu j^\mu = CE^\mu B_\mu, \tag{5}$$

where we have defined the electric and magnetic fields in the fluid rest frame, $E^\mu = F^{\mu\nu}u_\nu$, $B^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu F_{\alpha\beta}$. The right-hand sides of these equations take into account the fact that the external field performs work on the system, and the anomaly. Note that the rate of change of energy/momentum and particle number are $O(p)$ or $O(p^2)$ in our power counting, so the assumption of local thermal equilibrium is still valid.

The stress-energy tensor and the current are

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu} + \tau^{\mu\nu}, \tag{6}$$

$$j^\mu = nu^\mu + \nu^\mu, \tag{7}$$

where $\tau^{\mu\nu}$ and ν^μ are terms of order $O(p)$ which incorporate, in particular, dissipative effects. Following Landau and Lifshitz, we can always require $u_\mu\tau^{\mu\nu} = u_\mu\nu^\mu = 0$. We find $\tau^{\mu\nu}$ and ν^μ from the requirement of the existence of an entropy current s^μ with non-negative derivative, $\partial_\mu s^\mu \geq 0$. Transforming $u_\nu\partial_\mu T^{\mu\nu} + \mu\partial_\mu j^\mu$ using hydrodynamic equations and $\epsilon + P = Ts + \mu n$, we find

$$\partial_\mu\left(su^\mu - \frac{\mu}{T}\nu^\mu\right) = -\frac{1}{T}\partial_\mu u_\nu\tau^{\mu\nu} - \nu^\mu\left(\partial_\mu\frac{\mu}{T} - \frac{E_\mu}{T}\right) - C\frac{\mu}{T}EB. \tag{8}$$

In the standard treatment when the current is not anomalous, $C = 0$, this equation is interpreted as the equation of entropy production. The first-derivative parts of the energy tensor and the current have the following form:

$$\tau^{\mu\nu} = -\eta P^{\mu\alpha}P^{\nu\beta}(\partial_\alpha u_\beta + \partial_\beta u_\alpha) - \left(\zeta - \frac{2}{3}\eta\right)P^{\mu\nu}\partial u, \tag{9}$$

$$\nu^\mu = -\sigma TP^{\mu\nu}\partial_\nu\left(\frac{\mu}{T}\right) + \sigma E^\mu, \tag{10}$$

where $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$, and the entropy production rate is manifestly positive. However, in the presence of anomalies the last term in Eq. (8) can have either sign, and can overwhelm the other terms. Therefore, the hydrodynamic equations have to be modified.

The most general modification one can make is to add the following terms to the U(1) and entropy currents,

$$\nu^\mu = -\sigma TP^{\mu\nu}\partial_\nu\left(\frac{\mu}{T}\right) + \sigma E^\mu + \xi\omega^\mu + \xi_B B^\mu, \tag{11}$$

$$s^\mu = su^\mu - \frac{\mu}{T}\nu^\mu + D\omega^\mu + D_B B^\mu, \tag{12}$$

where ξ, ξ_B, D, and D_B are functions of T and μ. Now the entropy production $\partial_\mu s^\mu$ is a sum of many terms, including those containing ω^μ, B^μ, or other structures involving the Levi-Civita tensor. These terms are dangerous for the second law of thermodynamics because they can have either sign. In fact, it is possible to show that the requirement of positive entropy production cannot be satisfied for all initial conditions unless these terms vanishes. Using the following identities which follow from the ideal hydrodynamic equations,

$$\partial_\mu\omega^\mu = -\frac{2}{\epsilon + P}\omega^\mu(\partial_\mu P - nE_\mu), \tag{13}$$

$$\partial_\mu B^\mu = -2\omega E + \frac{1}{\epsilon + P}(-B\partial P + nEB), \tag{14}$$

one finds that the following four equations have to be satisfied:

$$\partial_\mu D - 2\frac{\partial_\mu P}{\epsilon + P}D - \xi\partial_\mu\frac{\mu}{T} = 0, \tag{15}$$

$$\partial_\mu D_B - \frac{\partial_\mu P}{\epsilon + P}D_B - \xi_B\partial_\mu\frac{\mu}{T} = 0, \tag{16}$$

$$\frac{2nD}{\epsilon + P} - 2D_B + \frac{\xi}{T} = 0, \tag{17}$$

$$\frac{nD_B}{\epsilon + P} + \frac{\xi_B}{T} - C\frac{\mu}{T} = 0. \tag{18}$$

To proceed further, we change variables from μ, T to a new pair of variables, $\bar{\mu} \equiv \mu/T$ and P. From $dP = sdT + nd\mu$, it is easy to derive

$$\left(\frac{\partial T}{\partial P}\right)_{\bar{\mu}} = \frac{T}{\epsilon + P}, \qquad \left(\frac{\partial T}{\partial \bar{\mu}}\right)_P = -\frac{nT^2}{\epsilon + P}. \tag{19}$$

Writing $\partial_i D = (\partial D/\partial P)\partial_i P + (\partial D/D\bar{\mu})\partial_i \bar{\mu}$, and noting that $\partial_i P$ and $\partial_i \bar{\mu}$ can be arbitrary, as they can be considered as initial condition on a time slice, Eq. (15) becomes two equations

$$-\xi + \frac{\partial D}{\partial \bar{\mu}} = 0, \qquad \frac{\partial D}{\partial P} - \frac{2}{\epsilon + P}D = 0. \tag{20}$$

Using Eq. (19), one finds that the most general solution to Eq. (20) is

$$D = T^2 d(\bar{\mu}), \qquad \xi = \frac{\partial}{\partial \bar{\mu}}[T^2 d(\bar{\mu})]_P, \tag{21}$$

where $d(\bar{\mu})$ is, for now, an arbitrary function of one variable. Equation (16) yields

$$D_B = T d_B(\bar{\mu}), \qquad \xi_B = \frac{\partial}{\partial \bar{\mu}}[T d_B(\bar{\mu})]_P, \tag{22}$$

where $d_B(\bar{\mu})$ is another function of $\bar{\mu}$. From Eqs. (17) and (18) we get

$$d_B(\bar{\mu}) = \frac{1}{2}d'(\bar{\mu}), \qquad d_B'(\bar{\mu}) - C\bar{\mu} = 0, \tag{23}$$

which can be integrated. We find

$$d_B(\bar{\mu}) = \frac{1}{2}C\bar{\mu}^2, \qquad d(\bar{\mu}) = \frac{1}{3}C\bar{\mu}^3. \tag{24}$$

So the new kinetic coefficients are

$$\xi = C\left(\mu^2 - \frac{2}{3}\frac{n\mu^3}{\epsilon + P}\right), \qquad \xi_B = C\left(\mu - \frac{1}{2}\frac{n\mu^2}{\epsilon + P}\right). \tag{25}$$

Extension to multiple charges.—It is easy to extend to the case of many charges. Here we consider only the U(1) charges that commute with each other. We denote the anomaly coefficients as C^{abc}, which is totally symmetric under permutation of indices and give the divergence of the gauge-invariant currents,

$$\partial_\mu j^{a\mu} = C^{abc}E^b B^c. \tag{26}$$

The constitutive equation is now

$$j^{a\mu} = n^a u^\mu + \cdots + \xi^a \omega^\mu + \xi_B^{ab} B^\mu, \tag{27}$$

where ξ^a and ξ_B^{ab} are new transport coefficients. The entropy current is now modified to

$$s^\mu = su^\mu - \frac{\mu^a}{T}\nu^a + D\omega^\mu + D_B^a B^{a\mu}. \tag{28}$$

Repeating the calculations of the previous section, we find that

$$D = \frac{1}{3}C^{abc}T^2\bar{\mu}^a\bar{\mu}^b\bar{\mu}^c, \qquad D_B^a = \frac{1}{2}C^{abc}T\bar{\mu}^b\bar{\mu}^c, \tag{29}$$

$$\xi^a = \frac{\partial}{\partial \bar{\mu}^a}D\bigg|_P, \qquad \xi_B^{ab} = \frac{\partial}{\partial \bar{\mu}^a}D_B^b\bigg|_P, \tag{30}$$

and, by using thermodynamic relations, one derives Eq. (4).

Gravity calculation.—The discussion above has been completely independent of details of the theory. We would like to check our formulas for the case when the kinetic coefficients can be calculated explicitly. In this Letter we use a holographic model as a testing ground for our predictions. Namely, we look at the theory described by the following 5D action,

$$S = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g_5}\bigg(R + 12 - F_{AB}F^{AB} + \frac{4\kappa}{3}\epsilon^{LABCD}A_L F_{AB}F_{CD}\bigg). \tag{31}$$

Here Latin indices A, B denote bulk 5D coordinates r, v, x, y, z, and Greek indices μ, $\nu \in \{v, x, y, z\}$ denote the boundary coordinates (v play the role of time on the boundary). The above action is a consistent truncation of type IIB supergravity Lagrangian on $AdS_5 \times S^5$ background with a cosmological constant $\Lambda = -6$ and the Chern-Simons parameter $\kappa = -1/(2\sqrt{3})$ [13,14]. In this case it describes $\mathcal{N} = 4$ supersymmetric Yang-Mills theory at strong coupling, where the U(1) charge corresponds to one particular subgroup of SO(6) internal symmetry. To keep the discussion general we will keep the κ coefficient unfixed, and treat Eq. (31) as the definition of our theory.

The field equations corresponding to (31) are

$$G_{AB} - 6g_{AB} + 2(F_{AC}F^C{}_B + \frac{1}{4}g_{AB}F^2) = 0, \tag{32}$$

$$\nabla_B F^{BA} + \kappa\epsilon^{ABCDE}F_{BC}F_{DE} = 0, \tag{33}$$

where g_{AB} is the 5D metric, $G_{AB} = R_{AB} - \frac{1}{2}g_{AB}R$ is the five dimensional Einstein tensor. The external gauge field and the current in the boundary theory are associated with the asymptotics of the A_μ near the boundary

$$A_\mu(r, x) = A_\mu(x) - \frac{2\pi G_5}{r^2}j_\mu(x). \tag{34}$$

From Eq. (33) one derives the relationship between the anomalies coefficient C and κ,

$$C = -\frac{2}{\pi G_5}\kappa. \tag{35}$$

These equations admit an AdS Reissner-Nordström black-brane solution. In Eddington-Finkelstein coordinates, it is

$$ds^2 = 2dvdr - r^2 f(r, m, q)dv^2 + r^2 d\vec{x}^2, \tag{36}$$

$$A = -\frac{\sqrt{3}q}{2r^2}dv, \tag{37}$$

where

$$f(r, m, q) = 1 - \frac{m}{r^4} + \frac{q^2}{r^6}. \tag{38}$$

The black brane is dual to a fluid at finite temperature T and chemical potential μ. The connection between the parameters of the metric and T and μ is

$$m = \frac{\pi^4 T^4}{2^4}(\gamma+1)^3(3\gamma-1), \qquad \gamma \equiv \sqrt{1+\frac{8\mu^2}{3\pi^2 T^2}}, \tag{39a}$$

$$q = \frac{2\mu}{\sqrt{3}}\frac{\pi^2 T^2}{4}(\gamma+1)^2. \tag{39b}$$

The equation of state of this fluid is

$$P(T, \mu) = \frac{m(T,\mu)}{16\pi G_5}. \tag{40}$$

In order to find the hydrodynamic equations, we use the method developed in Ref. [15]. We locally boost the Reissner-Nordström metric and consider the boost velocity u^μ, as well as the mass and charge of the black hole, as slowly-varying function of the black-brane coordinates, and also turn on a background gauge field A_μ^{bg}. To zeroth order, the background we obtain is

$$ds^2 = -2u_\mu dx^\mu dr + r^2(P_{\mu\nu} - f u_\mu u_\nu)dx^\mu dx^\nu, \tag{41}$$

$$A = \frac{\sqrt{3}q_0}{2r^2}u_\mu dx^\mu + A_\mu^{\mathrm{bg}} dx^\mu. \tag{42}$$

By iteration we construct the corrections proportional to first derivatives

$$g_{AB} = g_{AB}^{(0)} + g_{AB}^{(1)} + \ldots, \qquad A_M = A_M^{(0)} + A_M^{(1)} + \ldots, \tag{43}$$

requiring the solution to be regular at the horizon. The solution is then expanded around the boundary; ξ and ξ_B are read from the asymptotics of A_μ near the boundary. As the result, we find

$$\xi = -\frac{3q^2\kappa}{2\pi G_5 m}, \tag{44}$$

$$\xi_B = -\frac{\sqrt{3}(3R^4+m)q\kappa}{4\pi G_5 m R^2}, \tag{45}$$

where R is the radius of the horizon, $R = \frac{\pi}{2}T(\gamma+1)$. Both ξ and ξ_B originate from the Chern-Simons term in the 5D action (31), as evident from the presence of κ in their expressions. Equation (44) is consistent with previous results of Refs. [9,10], while Eq. (45) is a new result. It is straightforward to check that the result coincides with Eq. (25), computed using the equation of state (40) and the relationships (39).

Conclusion.—In this Letter we show that the relativistic hydrodynamic equations have to be modified in order to take into account effects of anomalies. At nonzero chemical potentials, we find a new effect of vorticity-induced current. Moreover, the kinetic coefficient characterizing this effect is completely fixed by the anomalies and the equations of state.

As evident from Eqs. (3) and (4), vorticity-induced currents appear only in the presence of chemical potentials. With no chemical potentials, vorticity does not induce current at all. We expect that the anomalous terms will play a role in noncentral heavy-ion collisions. One can draw a parallel with the "chiral magnetic effect," invoked to explain fluctuations of charge asymmetry in noncentral collisions [16,17]. They should also affect the hydrodynamic behavior of a dense and hot neutrino gas, or of the early Universe with a large lepton chemical potential.

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