PHYSICAL REVIEW LETTERS

Feynman Propagator for a Free Scalar Field on a Causal Set

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(Received 18 September 2009; published 29 October 2009)

The Feynman propagator for a free bosonic scalar field on the discrete spacetime of a causal set is presented. The formalism includes scalar field operators and a vacuum state which define a scalar quantum field theory on a causal set. This work can be viewed as a novel regularization of quantum field theory based on a Lorentz invariant discretization of spacetime.

The possibility that spacetime is fundamentally discrete presents one way to regulate the divergences in quantum field theory and the singularities in general relativity. Causal set theory provides a model of discrete spacetime in which spacetime events are represented by elements of a causal set—a locally finite, partially ordered set in which the partial order represents the causal relationships between events.

Causal sets can be considered as physical spacetimes in their own right [1] (in particular, in attempting to solve the problem of quantum gravity [2–4]) or they can be considered as merely Lorentz invariant discretizations of continuum spacetimes: Lorentzian random lattices. Here we develop a formalism for quantum field theory on a causal set which is applicable to both points of view (alternative approaches include [5,6]). The main result is the construction of the Feynman propagator for a real scalar field on a causal set. This work extends previous results for propagators on a causal set [7].

The ultimate aim would be to develop physically realistic quantum field theories on a causal set. Not only would one expect such field theories to be better defined than their continuum counterparts but one would hope to obtain new predictions for the behavior of matter which could provide evidence for spacetime discreteness.

A *causal set* (or *causet*) is a locally finite partial order, i.e., a pair $(\mathcal{C}, \preceq)$ where $\mathcal{C}$ is a set and $\preceq$ a relation on $\mathcal{C}$ which is (i) reflexive ($x \preceq x$), (ii) antisymmetric ($x \preceq y \preceq x \Rightarrow x = y$), (iii) transitive ($x \preceq y \preceq z \Rightarrow x \preceq z$), and (iv) locally finite ($|\{z' \in \mathcal{C} | x \preceq z' \preceq y\}| < \infty$) for all x, y, $z \in \mathcal{C}$. Here $|A|$ denotes the cardinality of a set A. We write $x \prec y$ if $x \preceq y$ and $x \neq y$.

The set $\mathcal{C}$ represents the set of spacetime events and the partial order $\preceq$ represents the causal order between pairs of events. If $x \preceq y$ we say "x is to the causal past of y". The causal relation of a Lorentzian manifold (without closed causal curves) satisfies conditions (i)–(iii). It is condition (iv) that enforces spacetime discreteness—each causal interval contains only a finite number of events.

A *sprinkling* is a way to generate a causal set from a d-dimensional Lorentzian manifold (M, g). Points are placed at random within M using a Poisson process (with density ρ) so the expected number of points in a region of d-volume V is ρV. This generates a causal set whose elements are the sprinkled points and whose partial order relation is "read off" from the manifold's causal relation restricted to the sprinkled points.

Here we shall only consider causal sets generated by sprinkling into d-dimensional Minkowski spacetime, $\mathbb{M}^d$. Such causal sets provide a discretization of $\mathbb{M}^d$ which, unlike a regular lattice, is Lorentz invariant ([4] Sec 1.5).

A *link* between u, $v \in \mathcal{C}$ (written $u \prec* v$) is a relation $u \prec v$ such that there exists no $w \in \mathcal{C}$ with $u \prec w \prec v$.

A *linear extension* of the causal set $(\mathcal{C}, \preceq)$ is a total order $(\mathcal{C}, \leq)$ which is consistent with the partial order. This means that $u \preceq v \Rightarrow u \leq v$ for all u, $v \in \mathcal{C}$.

Labeling the elements of a finite causal set (with p elements) $v_1, \ldots, v_p$ there are two $p \times p$ *adjacency matrices*: the *causal matrix* C and *link matrix* L defined by

$$C_{xy} := \begin{cases} 1 & \text{if } v_x \prec v_y \\ 0 & \text{otherwise,} \end{cases} \qquad L_{xy} := \begin{cases} 1 & \text{if } v_x \prec* v_y \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Free scalar quantum field theory in the continuum.—We briefly review the quantum field theory for a free real scalar field on $\mathbb{M}^d$ (following the sign conventions of [8]). The defining equation for a Green's function $G^{(d)}$ for the Klein-Gordon equation is

$$(\Box + m^2)G^{(d)}(x) = \delta^d(x). \quad (2)$$

Here $x = (x^0, \vec{x}) = (x^0, x^i)$ (for $i = 1, \ldots, d - 1$), m is the mass of the field, δ^d is the d-dimensional delta function and we choose units with $\hbar = c = 1$. The d'Alembertian is $\Box := \partial^2_{x^0} - \partial^2_{\vec{x}}$.

Three important Green's functions (or *propagators*) are the *retarded* G_R, *advanced* G_A and *Feynman* G_F. In $\mathbb{M}^2$ we have [7], ([9], p. 23)

$$G_R^{(2)}(x) := \theta(x^0)\theta(s^2)\frac{1}{2}J_0(ms), \quad (3)$$

$$G_A^{(2)}(x) := G_R^{(2)}(-x), \quad (4)$$

$$G_F^{(2)}(x) := \frac{1}{4}H_0^{(2)}(ms), \quad (5)$$

where $s = \sqrt{(x^0)^2 - (x^1)^2}$ for $(x^0)^2 \geq (x^1)^2$ and $s = -i\sqrt{(x^1)^2 - (x^0)^2}$ for $(x^0)^2 < (x^1)^2$. J_0 is a Bessel function of the first kind, $H_0^{(2)}$ is a Hankel function of the second kind and

$$\theta(\alpha) = \begin{cases} 1 & \text{if } \alpha \geq 0 \\ 0 & \text{if } \alpha < 0. \end{cases} \tag{6}$$

In $\mathbb{M}^4$ we have ([8] Appendix 1):

$$G_R^{(4)}(x) := \theta(x^0)\theta(s^2)\left(\frac{\delta(s^2)}{2\pi} - \frac{m}{4\pi s}J_1(ms)\right), \tag{7}$$

$$G_A^{(4)}(x) := G_R^{(4)}(-x), \tag{8}$$

$$G_F^{(4)}(x) := \frac{1}{4\pi}\delta(s^2) - \frac{m}{8\pi s}H_1^{(2)}(ms), \tag{9}$$

where $s = \sqrt{(x^0)^2 - \vec{x}^2}$ for $(x^0)^2 \geq \vec{x}^2$ and $s = -i\sqrt{\vec{x}^2 - (x^0)^2}$ for $(x^0)^2 < \vec{x}^2$. J_1 is a Bessel function of the first kind and $H_1^{(2)}$ is a Hankel function of the second kind and $\delta(s^2)$ is the delta function of s^2.

The Pauli-Jordan function is defined as [8]

$$\Delta(x) := G_R(x) - G_A(x). \tag{10}$$

We drop the dimension superscripts whenever we refer to arbitrary dimension.

Fields in the continuum.—A free real bosonic scalar field is represented by an algebra of field operators $\hat{\phi}(x)$ (which act on a Fock space F) satisfying the following conditions:

$$1. \quad (\Box + m^2)\hat{\phi}(x) = 0, \tag{11}$$

$$2. \quad \hat{\phi}(x) = \hat{\phi}(x)^\dagger, \tag{12}$$

$$3. \quad [\hat{\phi}(x), \hat{\phi}(y)] = i\Delta(y - x). \tag{13}$$

In addition there exists a Poincaré invariant vacuum state $|0\rangle \in F$. With the fields so defined the Feynman propagator is given by the vacuum expectation value of the time-ordered product of two field operators:

$$G_F(y - x) = i\langle 0|T\hat{\phi}(x)\hat{\phi}(y)|0\rangle. \tag{14}$$

The time ordering has time increasing from right to left.

Applying $(\Box_x + m^2)$ to the commutator in (13) gives

$$[(\Box_x + m^2)\hat{\phi}(x), \hat{\phi}(y)] = (\Box_x + m^2)i\Delta(y - x) = 0. \tag{15}$$

That is, even if only (12) and (13) hold, we have that $(\Box_x + m^2)\hat{\phi}(x)$ commutes with all the $\hat{\phi}(y)$ [10].

Free scalar quantum field theory on a causal set.—Let $(\mathcal{C}, \preceq)$ be a finite causal set with p elements $v_1, \ldots, v_p$ generated by sprinkling (with density ρ) into a finite causal interval in $\mathbb{M}^d$.

For sprinklings into $\mathbb{M}^2$ the $p \times p$ matrix

$$K_R^{(2)} := aC(I - abC)^{-1}, \quad a = \frac{1}{2}, \quad b = -\frac{m^2}{\rho}, \tag{16}$$

(where I is the identity matrix, C the causal matrix for the causal set and m is the mass of the field) is the appropriate definition of the retarded propagator [7].

For sprinklings into $\mathbb{M}^4$ the $p \times p$ matrix

$$K_R^{(4)} := aL(I - abL)^{-1}, \quad a = \frac{\sqrt{\rho}}{2\pi\sqrt{6}}, \quad b = -\frac{m^2}{\rho}, \tag{17}$$

(where L is the link matrix for the causal set) is the appropriate definition of the retarded propagator. Both these matrices were defined and studied in detail in [7].

Our goal now is to define a $p \times p$ matrix K_F to serve as the Feynman propagator on a causal set. We drop the dimension superscripts to be able to refer to sprinklings in either $\mathbb{M}^2$ or $\mathbb{M}^4$.

If K_R is the retarded propagator it follows that its transpose $K_A := K_R^T$ is the advanced propagator and the real matrix defined by

$$\Delta := K_R - K_A, \tag{18}$$

is the causal set analogue of the Pauli-Jordan function.

The matrix $i\Delta$ is skew-symmetric and Hermitian. These two properties ensure its rank is even [11] and its nonzero eigenvalues appear in real positive and negative pairs. In particular (if its rank is nonzero) there exist nonzero eigenvalues and normalized eigenvectors such that:

$$i\Delta u_i = \lambda_i u_i, \qquad i\Delta v_i = -\lambda_i v_i, \tag{19}$$

(with $\lambda_i > 0$) for $i = 1, \ldots, s$ where $2s$ is the rank of $i\Delta$.

These eigenvectors are each defined up to a phase factor and can be chosen such that $u_i = v_i^*$, $u_i^\dagger u_j = v_i^\dagger v_j = \delta_{ij}$, $u_i^\dagger v_j = 0$ for all $i, j = 1, \ldots, s$ (where z^* denotes the complex conjugate of z and $u^\dagger := (u^*)^T$ is the Hermitian conjugate of a column vector u).

It is useful to define the Hermitian $p \times p$ matrix

$$Q := \sum_{i=1}^{s} \lambda_i u_i u_i^\dagger, \tag{20}$$

such that $i\Delta = Q - Q^* = Q - Q^T$.

Fields on a causal set.—We now define an algebra of field operators $\hat{\phi}_x$ (acting on some Hilbert space H) to represent a free real bosonic scalar field on the causal set. For each causal set element v_x ($x = 1, \ldots, p$), we suppose there exists a field operator $\hat{\phi}_x$ such that

$$1. \quad \hat{\phi}_x = \hat{\phi}_x^\dagger, \tag{21}$$

$$2. \quad [\hat{\phi}_x, \hat{\phi}_y] = i\Delta_{xy}, \tag{22}$$

$$3. \quad i\Delta w = 0 \Rightarrow \sum_{x'=1}^{p} w_{x'}\hat{\phi}_{x'} = 0, \tag{23}$$

for $x, y = 1, \ldots, p$. The first two conditions are natural generalizations of the continuum case. The last condition ensures that any linear combination of field operators that commutes with all the field operators must be zero. By (15) this is the analogue of imposing the Klein-Gordon equation on the field operators.

From these field operators we can define new operators

$$\hat{a}_i := \sum_{x=1}^{p} (v_i)_x \hat{\phi}_x, \qquad \hat{a}_i^\dagger := \sum_{x=1}^{p} (u_i)_x \hat{\phi}_x. \qquad (24)$$

for $i = 1, \ldots, s$. They satisfy $(\hat{a}_i^\dagger)^\dagger = \hat{a}_i$ and

$$[\hat{a}_i, \hat{a}_j] = v_i^T i\Delta v_j = -\lambda_j u_i^\dagger v_j = 0, \qquad (25)$$

$$[\hat{a}_i^\dagger, \hat{a}_j^\dagger] = u_i^T i\Delta u_j = \lambda_j v_i^\dagger u_j = 0, \qquad (26)$$

$$[\hat{a}_i, \hat{a}_j^\dagger] = v_i^T i\Delta u_j = \lambda_j u_i^\dagger u_j = \lambda_j \delta_{ij}, \qquad (27)$$

and can be interpreted as (unnormalized) creation and annihilation operators.

The transformation can be inverted to give

$$\hat{\phi}_x = \sum_{i=1}^{s} (u_i)_x \hat{a}_i + (v_i)_x \hat{a}_i^\dagger \qquad (28)$$

[here it is important we imposed (23)].

We now define a vacuum state vector $|0\rangle \in H$ by the conditions that $\hat{a}_i |0\rangle = 0$ for all $i = 1, \ldots, s$ and $\langle 0|0\rangle = 1$. This allows us to recognize that H is the Fock space spanned by basis vectors $(\hat{a}_1^\dagger)^{n_1}(\hat{a}_2^\dagger)^{n_2} \cdots (\hat{a}_s^\dagger)^{n_s}|0\rangle$ for integers $n_i \geq 0$, $i = 1, \ldots, s$.

The two-point function can be evaluated as

$$\langle 0|\hat{\phi}_x \hat{\phi}_y|0\rangle = \sum_{i=1}^{s} \sum_{j=1}^{s} (u_i)_x (v_j)_y \langle 0|\hat{a}_i \hat{a}_j^\dagger|0\rangle$$

$$= \sum_{i=1}^{s} \sum_{j=1}^{s} (u_i)_x (v_j)_y \lambda_j \delta_{ij} = Q_{xy}. \qquad (29)$$

We can now define the Feynman propagator by analogy with (14). With time increasing from right to left we have

$$(K_F)_{xy} := i\langle 0|T\hat{\phi}_x \hat{\phi}_y|0\rangle$$
$$:= i(\bar{A}_{xy} Q_{yx} + \bar{A}_{yx} Q_{xy} + \delta_{xy} Q_{xy}), \qquad (30)$$

where $\bar{A}$ is the causal matrix for a linear extension of the causal set and δ_{xy} is the Kronecker delta. In general there are multiple different linear extensions which assign an arbitrary order to pairs of unrelated elements. This arbitrariness does not affect K_F because the field operators for unrelated elements commute.

Observing that $\bar{A}_{xy}(i\Delta_{xy}) = (iK_R)_{xy}$ we have

$$\bar{A}_{xy} Q_{yx} = \bar{A}_{xy}(Q_{xy} - i\Delta_{xy}) = \bar{A}_{xy} Q_{xy} - i(K_R)_{xy}. \qquad (31)$$

Substituting this into (30) gives an alternative form

$$K_F = K_R + iQ, \qquad (32)$$

since $\bar{A}_{xy} + \bar{A}_{yx} + \delta_{xy} = 1$ for all $x, y = 1, \ldots, p$.

Since $i\Delta = Q - Q^*$ the imaginary part of Q is $\Im(Q) = \Delta/2$. Combining this with (32) and looking at the real and imaginary parts of K_F gives

$$\Re(K_F) = K_R - \frac{\Delta}{2} = \frac{K_R + K_A}{2}, \qquad (33)$$

$$\Im(K_F) = \Re(Q). \qquad (34)$$

Comparison with the continuum.—The causal set propagators depend on the particular random causal set that is sprinkled. By calculating their average value for different sprinklings we can compare the causal set and continuum propagators. To do this, first fix a finite causal interval $\mathbb{A} \subset \mathbb{M}^d$. Pick two points $X, Y \in \mathbb{A}$. Sprinkle a finite causal set into $\mathbb{A}$ with density ρ. Almost surely this will not contain X and Y so add X and Y to it to obtain a finite causal set $(\mathcal{C}, \preceq)$. For definiteness, label the causal set element X as v_1 and Y as v_2.

We now calculate K_R and K_F for $(\mathcal{C}, \preceq)$ and look at $(K_R)_{12}$ and $(K_F)_{12}$, i.e., the propagator values for the pair (X, Y). Let $\mathbb{E}(K_R^{(d)}|X, Y, \mathbb{M}^d, \rho)$ denote the expected value of $(K_R^{(d)})_{12}$ [and $\mathbb{E}(K_F|X, Y, \mathbb{M}^d, \rho)$ the expected value of $(K_F)_{12}$] averaged over all causal sets sprinkled into $\mathbb{A} \subset \mathbb{M}^d$ (with X and Y added in the manner described and for a fixed density ρ). It was shown in [7] that

$$\mathbb{E}(K_R^{(2)}|X, Y, \mathbb{M}^2, \rho) = G_R^{(2)}(Y - X), \qquad (35)$$

$$\lim_{\rho \to \infty} \mathbb{E}(K_R^{(4)}|X, Y, \mathbb{M}^4, \rho) = G_R^{(4)}(Y - X). \qquad (36)$$

Using these and (33) we have

$$\Re(\mathbb{E}(K_F|X, Y, \mathbb{M}^2, \rho)) = \Re[G_F^{(2)}(Y - X)], \qquad (37)$$

$$\lim_{\rho \to \infty} \Re(\mathbb{E}(K_F|X, Y, \mathbb{M}^4, \rho)) = \Re[G_F^{(4)}(Y - X)]. \qquad (38)$$

That is, the real part of the expected value of K_F is correct for $\mathbb{M}^2$ and correct in the infinite density limit for $\mathbb{M}^4$ [12]. We can compare the imaginary parts of K_F and G_F through numerical simulations.

By using a computer to perform sprinkling into finite regions of $1 + 1$ and $3 + 1$ Minkowski spacetime we can compute K_R, K_A, $i\Delta$, Q and K_F for a range of sprinkling densities and field masses. Plotting the real and imaginary parts of K_F against the absolute value of the proper time for causally related and spacelike separated pairs of sprinkled points allows us to see if K_F agrees with G_F.

Simulations using standard linear algebra software on a desktop computer give clear results for the $1 + 1$ dimensional case for causal sets with as few as 600 elements (see Fig. 1). The agreement is very good provided $0 \ll m \ll \sqrt{\rho}$. There is disagreement between the imaginary parts of K_F and $G_F^{(2)}$ as we take the field mass to zero but this seems
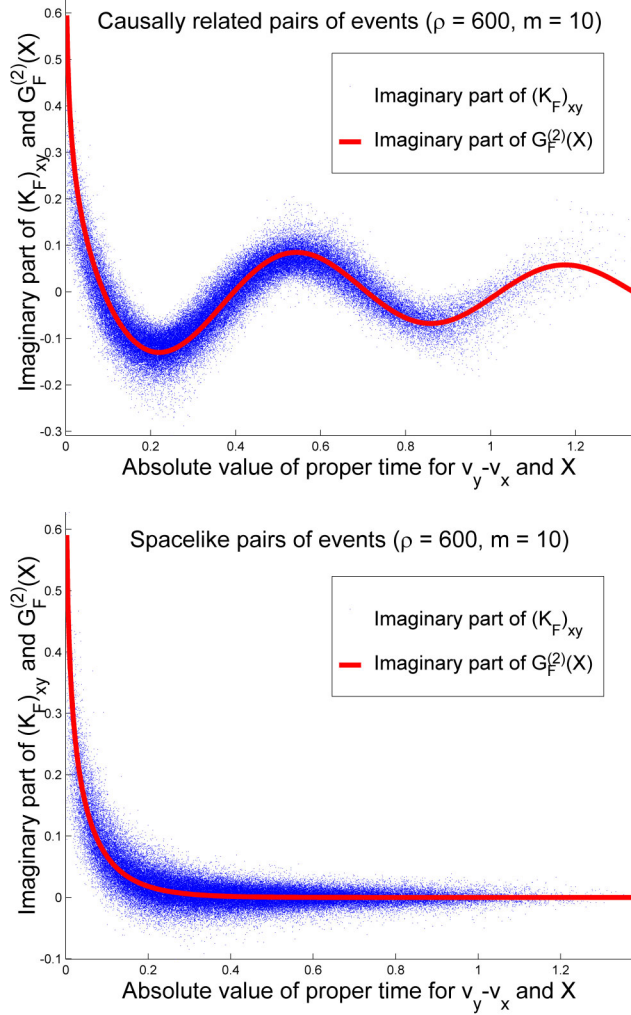
FIG. 1 (color online). Plots of $\Im((K_F)_{xy})$ for pairs of sprinkled points v_x and v_y in a *single* 600 element causal set generated by sprinkling into a unit-volume causal interval in $\mathbb{M}^2$ and $\Im(G_F^{(2)}(X))$ for $X \in \mathbb{M}^2$.

to be related to the lack of a massless limit of $\Im(G_F^{(2)})$. There are also small deviations due to the calculation being sprinkled into a *finite* region of Minkowski spacetime. These ''edge effects'' get smaller if the results are only plotted for pairs of points away from the edges of the region.

In $3+1$ dimensions the simulations are less clear. We expect good agreement with the continuum for large ρ [since the infinite density limit is needed in (36)] but larger sprinklings are needed to investigate this.

Discussion.—The formalism provokes interesting questions about the eigenvectors u_i, v_i. They seem to be related to linear combinations of positive and negative frequency plane wave solutions of the Klein-Gordon equation.

Plotting the eigenvector values as a function of position for causal sets sprinkled into $\mathbb{M}^2$ shows relatively smooth oscillatory behavior for the eigenvectors associated with large eigenvalues. They show some dependence on the shape of the region into which the points are placed but this dependence is lost when $\lambda_i u_i u_i^\dagger$ is summed in the Q and K_F matrices (details to appear elsewhere).

Extending K_F to infinite causal sets as well as further investigating the behavior of K_F for sprinklings into $\mathbb{M}^4$ as well as other conformally flat or curved spacetimes would be of significant interest. Investigating the variance of K_F for different sprinkling densities is also important. The propagator K_R (and therefore K_F) can be extended to nonsprinkled causal sets ([7] Sec 4.2). This ensures both K_R and K_F are relevant to a fundamental theory based only on causal sets with no reference to Lorentzian manifolds or sprinklings.

The work presented here opens the door to interacting scalar field theories, scalar field phenomenology, the evaluation of Feynman diagrams, *S*-matrices and scattering amplitudes on a causal set. Extending the quantum field theory to include interacting spinor and vector fields remains to be done—advances here could lead to physical predictions for matter on a causal set.

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