Manipulating Spatiotemporal Degrees of Freedom of Waves in Random Media

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We show that all the spatiotemporal degrees of freedom available in a complex medium can be harnessed and converted into spatial ones. This is demonstrated experimentally through an instantaneous spatial inversion, using broadband ultrasonic waves in a multiple scattering sample. We show theoretically that the inversion convergence is governed by the total number of degrees of freedom available in the medium for a fixed bandwidth and demonstrate experimentally its use for complex media investigation. We believe our approach has potential in sensing, imagery, focusing, and telecommunication.

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The principle of information transport within a medium for imaging, focusing, communication, or detection lies in wave propagation. Performances strongly depend on the behavior of the wave in the medium of interest. Within the last decades, complex and random media have gathered a lot of interest since physics associated with them offer more possibilities than their homogeneous counterparts. For example, it has been shown through information theory that the communication capacity between arrays of antennas increases with the complexity of the propagation medium [1]. Similarly, exploiting multiple scattering in a random medium allows one to focus waves in time and space on foci smaller than the Rayleigh criterion through the use of time reversal (TR) [2,3].

Most of these properties can be explained in terms of field correlations. A random medium, illuminated with a coherent source, generates a speckle pattern which can be characterized by a correlation length and a correlation frequency. The correlation length represents, at a given frequency, the typical dimension of a speckle grain, while the correlation frequency measures the minimal frequency change that leads to independent speckle patterns. From those two quantities and considering a given set of locations and a given spectral range, one can define a number of spatial and temporal degrees of freedom.

These degrees of freedom represent as many independent channels which can convey information. Most of the time, the temporal and spatial channels are considered independently. For instance coherent control of the optical field only exploits the spatial correlations of a medium [4]. Similarly, frequency hopping modulation techniques used in telecommunication take advantage of the temporal correlations only [5]. TR, which uses broadband waves and multiple sensors, has been proved to be a powerful principle for the study of waves in complex media [6], their use for focusing [2,7,8], telecommuni cation [9–11] or even sensing [12]. It takes advantage of all the spatiotemporal degrees of freedom, but does not distinguish between spatial and temporal ones, which results in a focusing that is equivalent spatially and temporally [6].

The question that arises is: can one make use of temporal degrees of freedom in order to obtain spatial information, or vice versa? Indeed, one typical problem in imaging or detection techniques is that the number of locations where one intends to focus, or equivalently the number of targets to be detected, must be smaller than the number of sensors used. This limitation comes from the fact that usual techniques require matrix inversion or pseudoinversion at a single frequency, thus being limited by the spatial degrees of freedom available.

In the present Letter, considering the simplest case of scalar acoustic waves in a two dimensional isotropic random medium, we show that it is indeed possible to make use of all the spatiotemporal degrees of freedom in order to perform a true spatial inversion. We demonstrate that all of them can be ‘‘converted’’ into spatial information, hence allowing spatial inversion, at a single time, on multiple locations using a single sensor. The method is based on an iterative TR scheme which does not require any matrix inversion. To support our findings we perform an experimental single sensor spatial inversion on multiple foci using ultrasound. Then, we study analytically and experimentally the limitations of the inversion procedure, and link them to the number of the degrees of freedom.

We consider a system composed of L sources in front of a time reversal mirror (TRM) with N_s emitters-receivers. The propagation between sources and the TRM can be described by the set of Green's functions $h_{ij}(t)$, $i \in [1; L]$, $j \in [1; N_s]$, relating the source i and the jth TRM's element. Using Fourier transform one can define the monochromatic propagation operator $\mathbf{H}(\omega)$, a $L \times N_s$ matrix.

In a typical TR experiment, time reversed Green's functions $h_{jk}(-t)$ are emitted from the TRM in order to focus a short pulse at location j. The result of this operation is that the waves emitted by the TRM converge towards position j and a short pulse is received at a specific time, which we define as $t = 0$. At any other times and locations, TR creates spatial and temporal sidelobes. Derode *et al.* [6] have shown that those sidelobes are equivalent and have a variance which is the amplitude of the focused pulse divided by the square root of the total number of degrees of freedom. Mathematically, emitting with the N_s mirror's transceivers, one obtains at any positions i and time t,

$$\mathbf{f}_{ij}(t) = \sum_{k=1}^{N_s} h_{ik}(t) * h_{jk}(-t), \tag{1}$$

where $*$ denotes temporal convolution. Matrix $\mathbf{f}$ can equivalently be written in the Fourier domain using matrix form

$$\mathbf{F}(\omega) = \mathbf{H}(\omega)\mathbf{H}^\dagger(\omega), \tag{2}$$

where $\dagger$ stands for the transpose conjugate $\mathbf{H}^\dagger = {}^t\mathbf{H}^*$. One can notice that $\mathbf{F}(\omega)$ is a square matrix of order L, independent from the TRM's size. From now, we only concentrate on the signals at time $t = 0$, that is to say, at the time of the maximum of the time reversed pulses. The values of the measured fields at this specific time simply writes

$$\mathbf{G} = \mathbf{f}(t = 0) = \int_\omega \mathbf{F}(\omega) d\omega. \tag{3}$$

Consequently, matrix $\mathbf{G}$, whose elements appear to be a sum of different frequency components, is a measure of the fields obtained after TR focusing on each of the L positions at time $t = 0$. The problem of interest is the inversion of this square matrix. Comparing to classical inverse problems, the inversion does not depend on the number of receivers constituting the TRM but on the spatiotemporal degrees of freedom of the medium. This is different from pseudoinversion of $\mathbf{H}(\omega)$ [13], which is an inversion frequency by frequency. Our approach drastically reduces the constraints of the inversion since it treats coherently all the frequencies at once.

The aim is to find a set of signals $\mathbf{E}$ that allows a perfect instantaneous spatial inversion. For that matter we have to cancel the spatial sidelobes, at time $t = 0$, on positions $j \neq i$, created by TR focusing on each position i (namely $\mathbf{G}_{ij}$). We start by normalizing the signals such that $\mathbf{G}$ has a unity diagonal (i.e. each focused pulse has a unit amplitude). The idea is to add to the initial signals that focus on position i, a sum of signals that focus on positions $j \neq i$ multiplied by the opposite of the sidelobes to be canceled ($-\mathbf{G}_{ij}$) This way we "erase" the sidelobes on positions $j \neq i$ created by focusing on position i. Naturally, this operation creates other lower sidelobes, and in order to completely invert the matrix we iterate the process until convergence, if possible. As a summary, the $n + 1$th emissions can be written in the Fourier domain as

$$\mathbf{E}^{(n+1)}(\omega) = 2\mathbf{E}^{(n)}(\omega) - {}^t\mathbf{G}^{(n)}\mathbf{E}^{(n)}(\omega). \tag{4}$$

Naturally, the algorithm starts with time reversal, i.e., emissions of $\mathbf{E}^{(1)} = \mathbf{H}^\dagger$. After propagation from the TRM toward the focal plane, the expression of the wave fields at each position $\mathbf{F}^{(n+1)}(\omega)$ and their value at the origin time $\mathbf{G}^{(n+1)}$ are

$$\begin{aligned} \mathbf{F}^{(n+1)}(\omega) &= \mathbf{F}^{(n)}(\omega)[2\mathbf{I}_L - \mathbf{G}^{(n)}], \\ \mathbf{G}^{(n+1)} &= \mathbf{G}^{(n)}[2\mathbf{I}_L - \mathbf{G}^{(n)}], \end{aligned} \tag{5}$$

where $\mathbf{I}_L$ is the identity matrix of size L. Again, it is important to keep in mind that $\mathbf{G}^{(n+1)}$ has no dependence in time (or in frequency) because it represents the fields at a specific time. Then, it is easy to demonstrate that, at the nth iteration, the instantaneous wave patterns obtained in the focal plane $\mathbf{G}^{(n)}$ follow a matrix geometric progression

$$\mathbf{G}^{(n)} = \mathbf{I}_L - [\mathbf{I}_L - \mathbf{G}]^{(2^{n-1})}. \tag{6}$$

As $\mathbf{G}$ is defined as a (continuous) sum of covariance matrices, it is a symmetric squared real matrix, similar to a diagonal matrix with positive eigenvalues. Therefore, according to Eq. (6), the suite $(\mathbf{G}^{(n)})_{n\in\mathbb{N}}$ must converge to the unit matrix $\mathbf{I}_L$ if the highest eigenvalue of $\mathbf{G}$, or equivalently its Euclidean matrix norm $\|\mathbf{G}\|_2$, is less than 2. Under this constraint, an instantaneous field that is non zero only at the focus can be designed. In addition, the exponent that governs the convergence 2^{n-1} gives a quick convergence and we get an almost perfect inversion after only a few iterations.

The Fresnel vectors representation can illustrate this spatial inversion. In the Fourier domain, $\mathbf{G}$ is the addition of various frequency components, each represented by a Fresnel vector. In a TR experiment, at the source position i, all spectral contributions have a zero phase and add constructively as $\int |\mathbf{H}_{ii}|^2 d\omega$. Outside the source, contributions of $\int \mathbf{H}_{ji}\mathbf{H}_{ii}^* d\omega$ are presumably uncorrelated for a complex medium and they add as a random walk [Fig. 1(a)]. After the iterative process, we have corrections on positions outside the source: the sum of Fresnel vectors cancel the initial Fresnel vectors, whereas at the focus they add randomly [Fig. 1(b)].

To test experimentally the potential of the iterative focusing inversion, we perform an experiment in the ultrasonic range. A single transducer element is used as TRM in front of a motorized plane transducer. They both work at a central frequency of $f_c = 1.5$ MHz. They are located on both sides of a multiple scattering medium, which consists in a random collection of parallel steel rods immersed in

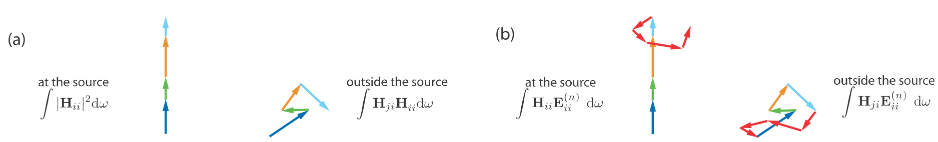

FIG. 1 (color online). Schematic representation of a broadband time reversal operation (a) and the iterative focusing (b). Each arrow (a Fresnel vector) represents a frequency component. (a) Frequency components add constructively at the source, and randomly outside. (b) Outside the source, the corrections (red arrows) cancel the initial field.

water. We first record a set of $L = 21$ by $N_s = 1$ Green's functions sampled at 32 MHz while the transducer emits a short pulse and is translated parallel to the array.

The iterative process is done numerically and after only five iterations the background level is lower than the computer resolution. The output of the algorithm is a set of L by N_s temporal signals $e_{ij}(t)$ that supposedly ensure an ideal focusing at time $t = 0$. As TR works as a matched filter, the signals $e_{ij}(t)$ must have more energy to ensure a same peak amplitude at focus. The imposed spatiotemporal constraints are small enough that the increase in energy at the emission is only 1.4 times that of TR.

These signals are then emitted by the TRM through the same scattering medium. The transducer that was previously used as a source is now a receiver, and it records the waveform generated in the focal plane. It was already shown earlier [6], that TR is a fairly robust operation that takes advantage of disorder to converge back to the source position and recover the original pulse duration. In our case, emissions are not the time reversed Green's functions and, even if they supposedly ensure a good focusing at the origin time, we must quantify the spatial and temporal focusing. Commonly, drawing the directivity pattern consists in keeping the maximum value in time of the wave field for each position j. The red dashed line on Fig. 2 shows the experimental focal spot using this definition. The emitted signals $e_{ij}(t)$ generate a wave field that is comparable to the one generated by time reversal (black dashed-dotted line). It means that the procedure does not reduce the temporal sidelobes: this is not a complete spatiotemporal inversion contrary to [13].

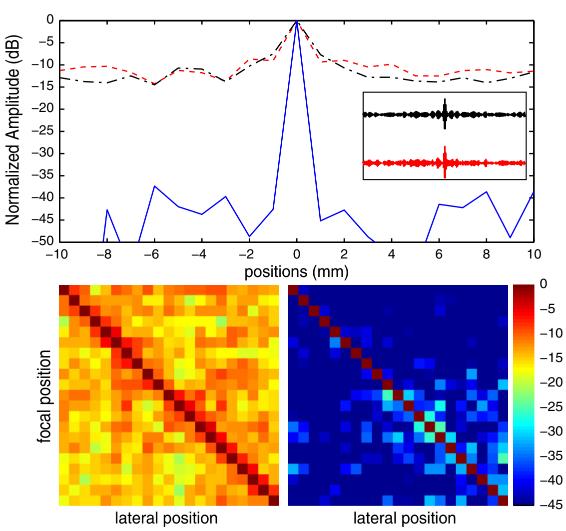

FIG. 2 (color online). (Top) Focal spot for time reversal (black dashed-dotted line) and emission of iterative signals (red dashed line) with one transducer source. The blue line represents the instantaneous spot with emission of the iterative signals. The focused pulses for TR and iterative signals emissions are represented in the inset. (Bottom) The 21 instantaneous focal spots for time reversal (left) and iterative signals (right): a true inversion over 21 points with only one source is shown.

To show the spatial inversion at time $t = 0$ (i.e., the arrival time of the peak), we plot in blue line the instantaneous focal spot. The improvement of the focusing is dramatic: the spatial side lobes level is reduced by nearly 30 dB. On a temporal point of view, the wave field generated by signals $e_{ij}(t)$ is similar to the one obtained with time reversal (see inset) except that the symmetry in time is broken. The same improvement is observed over the other positions: using a color map representation the inversion of matrix $\mathbf{G}$ is clearly demonstrated comparing the instantaneous focusing for the first iteration (i.e., TR) and after the convergence.

Now, the study of interest is the quantification of the method's limitations. We previously showed that the iterative process converges quickly towards identity if, and only if, the Euclidean norm $\|\mathbf{G}\|_2$ is less than 2. Off diagonal elements of $\mathbf{G}$ are estimations of cross-correlations between Green's functions. For simplicity we consider that they are uncorrelated due to the random medium, that is to say $\mathbf{H}_{ij}(\omega)$ is a Gaussian random complex variable with zero mean. By normalizing it in order to ensure an objective value of 1 at focus, its variance simply writes $\frac{1}{N_s}$, where N_s stands for the number of spatial degrees of freedom. Let us make a second approximation concerning the frequency dependence: the frequency spectrum is sampled with the correlation frequency $\delta\omega$, the minimal spectral length guaranteeing uncorrelated frequencies. Calling $\Delta\omega$ the bandwidth, the number of frequency, or temporal, degrees of freedom is $N_f = \frac{\Delta\omega}{\delta\omega}$. Thus, elements of matrix $\mathbf{G}$ with our approximations simply write

$$\mathbf{G}_{ij} = \frac{1}{N_f}\sum_{k=1}^{N_f}\sum_{n=1}^{N_s}\mathbf{H}_{in}^{\omega_k}\mathbf{H}_{jn}^{\omega_k *},$$

$$\langle \mathbf{H}_{in}^{\omega_k}\mathbf{H}_{im}^{\omega_l *}\rangle = \underbrace{\frac{1}{N_s}\delta_{nm}}_{\text{spatial hypothesis}} \quad \underbrace{\delta_{kl}}_{\text{frequency hypothesis}}, \qquad (7)$$

where δ_{ij} is the Kronecker symbol. We then calculate the mean and the variance of $\mathbf{G}_{ij}$ according to our hypothesis using momentum theorem:

$$\langle \mathbf{G}_{ij}\rangle = \delta_{ij}, \qquad \mathrm{var}[\mathbf{G}_{ij}] = \frac{1}{N_s N_f} \qquad (8)$$

So, matrix $\mathbf{G}$ can be seen as a sum of the L dimensional identity matrix $\mathbf{I}_L$ and a random squared matrix $\mathbf{R}$ with variance of $\frac{1}{N_s N_f}$. This permits us to overvalue its norm

$$\|\mathbf{G}\|_2 = \|\mathbf{I}_L + \mathbf{R}\|_2 \leq 1 + \|\mathbf{R}\|_2. \qquad (9)$$

Introducing results of Eqs. (8) into Eq. (9) allows to define a convergence criterion for the iterative process. This is a condition over the L potential uncoupled focusing positions versus the number $N = N_s N_f$ of spatiotemporal degrees of freedom

$$L \leq N = N_s N_f. \qquad (10)$$

Physically, this equation has a pretty simple meaning. When focusing on 1 position with TR, a background sidelobes level of $1/N$ is obtained. Now if one wants to focus on L positions, the sidelobes add incoherently, and their average amplitude is multiplied by $\sqrt{L}$. Thus, Eq. (10) states that the sidelobes have to be lower than the amplitude of the initial focused pulse in order to be iteratively eliminated.

One can notice that N is equal to the product of each type of degrees of freedom. Thus, in the monochromatic case ($N_f = 1$) where time reversal is equivalent to phase conjugation, the inversion should be possible only if the number of sources N_s is at least equal to the number of positions. This is why in monochromatic methods [14] or in the pseudoinversion processes [13] authors conclude to a number of emitters higher than the number of uncorrelated focal spots. Assuming that L and N_s are large enough, Sprik *et al.* [15] demonstrated that eigenvalues distribution of $\mathbf{F}(\omega)$ (i.e. in the monochromatic case) follows a Marcenko-Pastur law, where the highest eigenvalue is related to $\frac{L}{N_s}$. After integration over the whole bandwidth, the largest eigenvalue of $\mathbf{G}$ would be related to $\frac{L}{N_s N_f}$. Here, the instantaneous field can be inverted over L distinct positions even with a number of sources N_s equal to 1, provided that the number of uncorrelated frequencies N_f in the bandwidth compensates for the lack of sources.

According to this calculation, another interesting result is that the Euclidean norm of matrix $\mathbf{G}$ gives us information about the medium. Actually, if the highest eigenvalue of $\mathbf{G}$ is lower than 2, the convergence criterion is verified and the number of degrees of freedom is high enough to ensure inversion feasibility. For a given number of elements N_s, decreasing the available bandwidth of signals, a minimum value is reached when the inversion no longer converges:

$$\Delta\omega_{\min} = \frac{L\delta\omega}{N_s}. \tag{11}$$

The Wiener-Khinchin theorem states that the spectral correlation function is the Fourier transform of the "time of flight" distribution. In other words, if ΔT stands for the typical duration of the transmitted energy $|h(t)|^2$, the equality $\delta\omega = \frac{1}{\Delta T}$ is verified. Then, in a multiple scattering medium, in the diffusive approximation, this typical spreading time is equal to the so called Thouless time $\tau_{\text{thouless}} = \frac{l_0^2}{D}$, where l_0 is the medium's thickness and D is the diffusion coefficient. For isotropic scatterers, D is related to the elastic mean free path l_e [16] by $D = \frac{c_0 l_e}{d}$, where c_0 is the average speed of sound, and d the problem's dimension (here $d = 2$). Eventually, we have the equality

$$l_e(\omega) = \frac{2{l_0}^2 N_s \Delta\omega_{\min}(\omega)}{c_0 L}. \tag{12}$$

Thus, using the same set of Green's functions as in the previous experiment, we computationally determine (by dichotomy) the minimum bandwidth value allowing the convergence, as a function of the central frequency. Using experimental parameters, that is to say $l_0 = 3$ cm, $L = 21$, $c_0 = 1524\ \text{m}\cdot\text{s}^{-1}$ and $N_s = 1$, we get the evolution of the average elastic mean free path as a function of excitation's central frequency with one set of 21 Green's functions (Fig. 3). The reconstruction clearly reveals a resonant frequency around 2.75 MHz that corresponds to an elastic cross section peak using aluminum cylinder with 0.4 mm diameter as a scatterer [17].

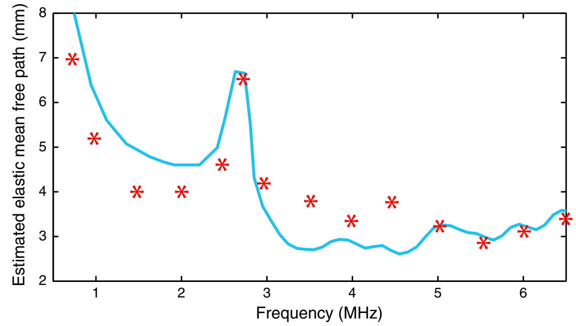

FIG. 3 (color online). Evolution with frequency of the reconstructed elastic mean free path. The reconstruction is done, according to Eq. (12), using convergence criterion. The red asterisks represent a theoretical prediction for a Gaussian distribution of radii [17] (mean radius 0.4 mm, standard deviation 8 mm).

In conclusion, we have shown how one can manipulate spatiotemporal degrees of freedom of waves in a random medium through a spatial inversion. Experiments done in the ultrasonic range with a strongly scattering medium confirmed the feasibility of such a method. We have found a convergence criterion and used it to investigate the statistical properties of the medium. We believe that this work can be extended to polarized waves and anisotropic random media, and will have applications within the fields of imaging, detection in scattering media, sensing techniques, and telecommunications.

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