Constraining the Energy-Momentum Dispersion Relation
with Planck-Scale Sensitivity Using Cold Atoms

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We use the results of ultraprecise cold-atom-recoil experiments to constrain the form of the energy-momentum dispersion relation, a structure that is expected to be modified in several quantum-gravity approaches. Our strategy of analysis applies to the nonrelativistic (small speeds) limit of the dispersion relation, and is therefore complementary to an analogous ongoing effort of investigation of the dispersion relation in the ultrarelativistic regime using observations in astrophysics. For the leading correction in the nonrelativistic limit the exceptional sensitivity of cold-atom-recoil experiments remarkably allows us to set a limit within a single order of magnitude of the desired Planck-scale level, thereby providing the first example of Planck-scale sensitivity in the study of the dispersion relation in controlled laboratory experiments.

Gaining experimental insight on the quantum-gravity realm is very challenging because most effects are expected to occur on the ultrahigh "Planck scale" M_P ($\simeq 1.2 \times 10^{28}$ eV), and leave only minuscule traces on processes we can access experimentally. But thanks to a large and determined effort made over the last decade [1–9] we do have now at least a few research lines in "quantum-gravity phenomenology" [10] where it is established that quantum properties of gravity and/or spacetime structure could be investigated with the desired Planck-scale sensitivity. Previously, progress in this direction had been obstructed by the extreme mathematical complexity of the most promising theories of quantum gravity, resulting in a debate on quantum gravity that was confined at the level of comparison of mathematical and conceptual features, without the ability to control the mathematics well enough to obtain robust derivations of the physical implications of the different scenarios.

At least for one aspect of the quantum-gravity problem, the one that concerns the possibility that spacetime itself might have to be quantized, the nature of the debate started to change in the second half of the 1990s when it was established that a general implication of spacetime quantization is a modification of the classical-spacetime "dispersion" relation between energy E and (modulus of) momentum p of a microscopic particle with mass m. In the nonrelativistic limit ($p \ll m$), which is here of interest, this dispersion relation should take the form

$$E \simeq m + \frac{p^2}{2m} + \frac{1}{2M_P}\left(\xi_1 mp + \xi_2 p^2 + \xi_3 \frac{p^3}{m}\right), \quad (1)$$

working in units with speed-of-light constant set to 1, and including only terms at leading order in the (inverse of the) Planck scale.

The model-dependent dimensionless parameters ξ_1, ξ_2, ξ_3 should (when different from zero) have values roughly of order one, so that indeed the new effects are introduced in some neighborhood of the Planck scale. Evidence that at least some of these parameters should be nonzero has been found most notably in loop quantum gravity [5,11,12], and, in particular, the framework introduced in Refs. [5,13], which was inspired by loop quantum gravity, produces a term linear in p in the nonrelativistic limit (the effect here parametrized by ξ_1). Other definite proposals for the parameters ξ_1, ξ_2, ξ_3 have emerged [14–17] from the quantum-gravity approach based on "noncommutative geometry", and the associated research area that contemplates deformations of special relativity such that one could have an observer-independent maximum value of frequency or minimum value of wavelength. The two most studied candidates are the ones first introduced in Ref. [18], whose leading-order form is

$$E = \sqrt{m^2 + p^2} - \frac{\eta}{M_P}\frac{p^2}{2} \quad (2)$$

and the one first introduced in Ref. [19], whose leading-order form is

$$E = \sqrt{m^2 + p^2} + \frac{\eta}{M_P}\left(\frac{m^3}{\sqrt{m^2 + p^2}} - m^2\right). \quad (3)$$

Interestingly both of these scenarios from the side of noncommutative geometry have the same behavior in the nonrelativistic limit, dominated by a p^2/M_P term of the type here parametrized with ξ_2. But in the ultrarelativistic limit they have very different behavior, and this will be of interest in a later part of our analysis.

In addition to these examples where something definite is expected for the parameters here of interest, which characterize the dispersion relation in the nonrelativistic limit, there is also a quantum-gravity literature providing motivation for studies of the dispersion relation from a broader perspective, but often within formalisms that are not understood well enough to establish the functional dependence of the correction on momentum. Nonetheless, many authors (see, e.g., Ref. [12] and references therein) have argued that our best chance of having a first level of experimental characterization of the quantum-gravity realm is through attempts to gain insight on the parameters of the dispersion relation.

Unfortunately, as usual in quantum-gravity research, the theoretically favored range of values of the parameters of the dispersion relation translates into a range of possible magnitudes of the effects that is extremely challenging. If the Planck scale is the characteristic scale of quantum-gravity effects then one expects that parameters such as ξ_1, ξ_2, ξ_3 should indeed take (positive or negative) values that are close to 1, and then, as a result of the overall factor $1/M_P$, the effects are terribly small [10]. Some recent semiheuristic renormalization-group arguments (see, e.g., Refs. [10,20] and references therein), have encouraged the intuition that the quantum-gravity scale might be somewhat smaller than the Planck scale, plausibly even 3 orders of magnitude smaller (so that it could coincide [20] with the "grand unification scale" which appears to be relevant in particle physics). This would correspond to an estimate of parameters such as ξ_1, ξ_2, ξ_3 plausibly as "high" as 10^3, but usually even with this possible gain of 3 orders of magnitude any hope of detectability remains extremely distant.

It was therefore rather exciting for many quantum-gravity researchers when it started to emerge that certain observations in astrophysics could provide "Planck-scale sensitivity" for some quantum-gravity scenarios [1–3,6,7]. However, these studies, which are presently being conducted at the Fermi Space Telescope [21], only establish meaningful bounds on scenarios with relatively strong ultrarelativistic corrections, such as the proposal of Ref. [18] [Eq. (2)] which produces a term of order p^2/M_P in the ultrarelativistic regime. But, for example, in the ultrarelativistic limit of the models of Ref. [19] [Eq. (3)] and Ref. [5] the effects are too small to matter.

Our main objective here is to show that cold-atom experiments can be used to establish meaningful bounds on the parameters ξ_1 and ξ_2 that characterize the nonrelativistic limit of the dispersion relation, and to discuss the relevance of this result particularly for the scenarios first proposed in Refs. [5,19]. The ultrahigh levels of accuracy [22,23] achievable with atom interferometry have been already exploited extensively in many areas of physics, including precision measurements of gravity [24], gravity gradients [25], and rotation of the Earth [26], and also tests of Einstein's weak equivalence principle [24,27], tests of

Newton's law at short distances [28], and measurements of fundamental physical constants [29,30]. Clearly, for our purposes, it is very significant that these remarkable accuracy levels have been reached in studies of nonrelativistic atoms.

The measurement strategy we here propose is applicable to measurements of the "recoil frequency" of atoms with experimental setups involving one or more "two-photon Raman transitions" [24,31,32]. Let us initially set aside the possibility of Planck-scale effects, and discuss the recoil of an atom in a two-photon Raman transition from the perspective adopted in Ref. [32], which provides a convenient starting point for the Planck-scale generalization we shall discuss later. One can impart momentum to an atom through a process involving absorption of a photon of frequency ν and (stimulated [24,31,32]) emission, in the opposite direction, of a photon of frequency ν'. The frequency ν is computed taking into account a resonance frequency ν_* of the atom and the momentum the atom acquires, recoiling upon absorption of the photon: $\nu \simeq \nu_* + (h\nu_* + p)^2/(2m) - p^2/(2m)$, where m is the mass of the atom (e.g., $m_{Cs} \simeq 124$ GeV for cesium), and p its initial momentum. The emission of the photon of frequency ν' must be such to deexcite the atom and impart to it additional momentum: $\nu' + (2h\nu_* + p)^2/(2m) \simeq \nu_* + (h\nu_* + p)^2/(2m)$. Through this analysis one establishes that by measuring $\Delta\nu \equiv \nu - \nu'$, in cases (not uncommon) where ν_* and p can be accurately determined, one actually measures h/m for the atoms

$$\frac{\Delta\nu}{2\nu_*(\nu_* + p/h)} = \frac{h}{m}. \qquad (4)$$

This result has been confirmed experimentally with remarkable accuracy. A powerful way to illustrate this success is provided by comparing the results of atom-recoil measurements of $\Delta\nu/[\nu_*(\nu_* + p/h)]$ and of measurements [33] of α^2, the square of the fine structure constant. α^2 can be expressed in terms of the mass m of any given particle [32] through the Rydberg constant, R_∞, and the mass of the electron, m_e, in the following way [32]: $\alpha^2 = 2R_\infty \frac{m}{m_e} \frac{h}{m}$. Therefore according to Eq. (4) one should have

$$\frac{\Delta\nu}{2\nu_*(\nu_* + p/h)} = \frac{\alpha^2}{2R_\infty} \frac{m_e}{m_u} \frac{m_u}{m}, \qquad (5)$$

where m_u is the atomic mass unit and m is the mass of the atoms used in measuring $\Delta\nu/[\nu_*(\nu_* + p/h)]$. The outcomes of atom-recoil measurements, such as the ones with cesium reported in Ref. [32], are consistent with Eq. (5) with the accuracy of a few parts in 10^9.

The fact that Eq. (4) has been verified to such a high degree of accuracy proves to be very valuable for our purposes as we find that modifications of the dispersion relation require a modification of Eq. (4). Our derivation can be summarized briefly by observing that the logical steps described above for the derivation of Eq. (4) establish

the following relationship

$$\Delta\nu \simeq E(p + h\nu + h\nu') - E(p) \simeq E(2h\nu_* + p) - E(p),$$
(6)

and therefore Planck-scale modifications of the dispersion relation, parametrized in Eq. (1), would affect $\Delta\nu$ through the modification of $E(2h\nu_* + p) - E(p)$, which compares the energy of the atom when it carries momentum p and when it carries momentum $p + 2h\nu_*$.

Since our main objective here is to expose sensitivity to a meaningful range of values of the parameter ξ_1, let us focus on the Planck-scale corrections with coefficient ξ_1. In this case the relation (4) is replaced by

$$\Delta\nu \simeq \frac{2\nu_*(h\nu_* + p)}{m} + \xi_1 \frac{m}{M_P} \nu_*,$$
(7)

and in turn in place of Eq. (5) one has

$$\frac{\Delta\nu}{2\nu_*(\nu_* + p/h)}\left[1 - \xi_1\left(\frac{m}{2M_P}\right)\left(\frac{m}{h\nu_* + p}\right)\right] = \frac{\alpha^2}{2R_\infty}\frac{m_e}{m_u}\frac{m_u}{m}.$$

We have arranged the left-hand side of this equation placing emphasis on the fact that our quantum-gravity correction is as usual penalized by the inevitable Planck-scale suppression (the ultrasmall factor m/M_P), but in this specific context it also receives a sizeable boost by the large hierarchy of energy scales $m/(h\nu_* + p)$, which in typical experiments of the type here of interest can be [24,31,32] of order $\sim 10^9$.

This turns out to be just enough to provide the desired "Planck-scale sensitivity": one easily finds that in light of our result the mentioned cesium-atom-recoil measurements the rubidium-atom-recoil measurements reported in Ref. [34] determine $\Delta\nu/[2\nu_*(\nu_* + p/h)]$ with accuracy comparable to the cesium experiments of Ref. [32]. However, in the setup of Ref. [34] the rubidium atoms had momentum p significantly higher than for the cesium atoms in Ref. [32], and, as a consequence of the specific dependence on p of our result, it turns out that the cesium measurements lead to a significantly more stringent limit on ξ_1 than the rubidium measurements.) reported in Ref. [32], also exploiting the high precision of a determination of α^2 recently obtained from electron-anomaly measurements [33], allow us to determine that $\xi_1 = -1.8 \pm 2.1$.

From this we derive the main result we are here reporting, which is the bound $-6.0 < \xi_1 < 2.4$, established at the 95% confidence level. This shows that the cold-atom experiments we here considered can be described as the first example of controlled laboratory experiments probing the form of the dispersion relation (at least in one of the directions of interest) with sensitivity that is meaningful from a Planck-scale perspective. We are actually already excluding a very substantial portion of the range of values of ξ_1 that could be natural from a quantum-gravity perspective, which, for reasons we briefly revisited above,

goes from $|\xi_1| \sim 1$ to $|\xi_1| \sim 10^3$. Our result leaves open only the possibility of a value of ξ_1 that lies rather close to the bottom end of the range that would be admissible from a quantum-gravity perspective.

Of course, studies of possible modifications of the dispersion relation are also of interest for the community involved in tests of Lorentz symmetry from a broader fundamental-physics perspective. And our bound on the parameter ξ_1 is also relevant for a class of modifications of the dispersion relation that has been studied from this broader perspective, by introducing a parameter λ such that $E^2 = m^2 + p^2 + 2\lambda p$. For this framework the previous reference limit was established in Ref. [35], which considered various strategies for obtaining bounds at the level $\lambda < 10$ eV. Taking into account that from $E^2 = m^2 + p^2 + 2\lambda p$ it follows that in the nonrelativistic limit $E = m + p^2/(2m) + \lambda p/m$, one easily finds that our parametrization and the parametrization of Ref. [35] are related by $\xi_1 m/M_P \equiv 2\lambda/m$. And our bound on ξ_1 amounts to the bound -3.7×10^{-6} eV $< \lambda < 1.5 \times 10^{-6}$ eV. From this perspective one should therefore observe that the remarkable accuracy of cold-atom experiments allowed us to improve on the previous best limit on λ by more than 6 orders of magnitude!

While our main results concern indeed the parameter ξ_1, we find appropriate to also briefly discuss the implications of cold-atom studies for the term with coefficient ξ_2. As mentioned, the term with coefficient ξ_2 in the nonrelativistic limit is a common feature of the two quantum-gravity-inpired proposals here characterized in Eqs. (2) and (3). Let us notice that the same behavior in the nonrelativistic limit is also found in the model of Ref. [36], whose proposal was not motivated by quantum gravity but has been much studied from the broader Lorentz-symmetry-test perspective. Interestingly, for these 3 models with the same dependence on momentum of the correction to energy in the nonrelativistic limit one finds completely different consequences in the ultrarelativistic regime. For the model of Eq. (2) the leading ultrarelativistic correction to energy has behavior p^2/M_P and can be tightly constrained in astrophysics [1–3]. And for the model of Ref. [36], whose leading ultrarelativistic correction to energy is instead linear in momentum, a similar strategy allows to set stringent limits using astrophysics data [36,37]. But for the third of these possibilities, the one of Ref. [19] [Eq. (3)], the leading correction to energy in the ultrarelativistic limit is only of magnitude $m^3/(pM_P)$ and cannot be significantly bounded in astrophysics. The effort of constraining the parameter ξ_2 in the nonrelativistic limit is not a top priority for the scenarios of Ref. [18] [Eq. (2)] and Ref. [36], since those scenarios can be even more tightly constrained studying their ultrarelativistic behavior, but on the contrary for the scenario of Ref. [19] [Eq. (3)], the only way to establish meaningful bounds is by investigating the nonrelativistic limit.

Following the same steps of the analysis we performed above for the correction term with coefficient ξ_1, it is easy to verify that the correction term with coefficient ξ_2 would produce the following modification of Eq. (5):

$$\frac{\Delta\nu}{2\nu_*(\nu_* + p/h)}\left[1 - \xi_2\frac{m}{M_P}\right] = \frac{\alpha_*^2}{2R_\infty}\frac{m_e}{m_u}\frac{m_u}{m}. \quad (8)$$

And in this case the experimental results reported in Ref. [32] allow us to establish that $-3.8\times10^9 < \xi_2 < 1.5\times10^9$. This bound is still some 6 orders of magnitude above even the most optimistic quantum-gravity estimates. But it is a bound that still carries some significance in the broader realm of Lorentz-symmetry investigations. According to standard quantum-spacetime arguments, bounds on parameters such as ξ_2 at the level of $|\xi_2| < 10^9$ amount to probing spacetime structure down to length scales of order 10^{-26} m ($\sim\xi_2 h/M_P$), and, while this is not enough for quantum gravity according to the prevailing consensus, still represents remarkably short distance scales from a broader perspective.

Moreover our limit on ξ_2 at the level $|\xi_2| \lesssim 10^9$ indeed also amounts to the best limit on the scenario for deformation of Lorentz symmetry introduced in Ref. [19], since in the nonrelativistic limit the parameter η of Eq. (3) is related to ξ_2 by $\xi_2 = 4\eta$. Previous attempts to constrain the parameter η of Eq. (3) had focused on the ultrarelativistic limit of Eq. (3), and did not go beyond [38,39] sensitivities at the level $|\eta| \lesssim 10^{24}$.

In light of the remarkable pace of improvement of cold-atom experiments over the last 20 years, we expect that the sensitivities here established might be improved upon in the near future. This will most likely translate into more stringent bounds, but, particularly considering the values of ξ_1 being probed, should also be viewed as a (slim but valuable) chance for a striking discovery. We therefore feel that our analysis should motivate a vigorous effort on the quantum-gravity side aimed at overcoming the mentioned technical difficulties that presently obstruct the derivation of more detailed quantitative predictions in some of the relevant theoretical frameworks.

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