

# Unconditional Security of Single-Photon Differential Phase Shift Quantum Key Distribution

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In this Letter, we prove the unconditional security of the single-photon differential phase shift quantum key distribution (DPS-QKD) protocol, based on the conversion to an equivalent entanglement-based protocol. We estimate the upper bound of the phase error rate from the bit error rate, and show that the DPS-QKD protocol can generate an unconditionally secure key when the bit error rate is not greater than 4.12%. This proof is the first step to the unconditional security proof of a coherent state DPS-QKD.

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Quantum key distribution (QKD) protocols are one of the most important applications of quantum information theory. Great effort has been devoted to proving the unconditional security of these protocols through noisy channels. The first QKD protocol, named the Bennett-Brassard 1984 (BB84) protocol [1], has been proven to be unconditionally secure [2,3].

In 2002, Inoue *et al.* proposed the differential phase shift quantum key distribution (DPS-QKD) protocol [4], where Alice encodes the key bits by preparing the relative phase shifts between two consecutive pulses in 0 or  $\pi$  and Bob employs a one-bit delay Mach-Zehnder (M-Z) interferometer to retrieve the key from the phase shifts. The advantages of DPS-QKD mainly lie in its simple and robust experimental implementation. Only one measurement basis is involved in the protocol, and thus the experiment requires minimum setup, namely, one source and two detectors. Also, for the BB84 protocol, Alice should generate a random base string to encode the key and Bob also needs to randomly select his measurement bases. However, in DPS-QKD, they do not need to perform these two steps. Furthermore, DPS-QKD utilizes the relative phases of the pulses which are not affected by the birefringence in fibers. Finally, by using coherent sources, DPS-QKD is secure against photon-number-splitting attack, because they can be detected [5], while the BB84 protocol with coherent sources requires intensity modulators to generate decoy states to prevent such an attack [6]. Thanks to these simplicities, experiments over long distances and with high bit rates have been performed [7,8]. On the other hand, whether DPS-QKD is unconditionally secure is an important problem both from the practical and theoretical aspects. So far, the security against limited attacks such as the general attack for individual photons [9] and the so-called sequential attack [10] have been analyzed.

In this Letter, as the first important step towards the security proof of coherent state DPS-QKD, we present the proof of the unconditional security of DPS-QKD with

a single-photon source against the most general attacks. This proof gives insight to the underlying security properties of DPS-QKD. By analyzing an equivalent entanglement-based DPS-QKD, we find that the phase error rate of each time slot can be upper bounded by the bit error rate of the same time slot and its adjacent time slots. Thus, the unconditional security is achieved by performing privacy amplification based on the upper bound of the phase error rate. Thanks to the equivalence, we can apply the results to the prepare-and-measure DPS-QKD.

Before constructing the entanglement-based protocol, we define the encoded states in the prepare-and-measure DPS-QKD. With the single-photon source, Alice splits the single-photon wave packet into  $n$  pulses with identical amplitudes to form a block. Particularly, the state before encoding the secret key is  $|\phi_0\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^n a_k^\dagger |\text{vac}\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^n |D_k\rangle$ , where  $a_k^\dagger$  is the creation operator of the pulse in the  $k$ th slot and  $|D_k\rangle = a_k^\dagger |\text{vac}\rangle$ . Then following the proposal of DPS-QKD, Alice encodes an  $(n-1)$ -bit random secret key into this block. For an  $(n-1)$ -bit random but fixed integer  $j$ , we express its  $(n-1)$ -bit binary format as  $(j_1 j_2 \cdots j_{n-2} j_{n-1})_2$ . Then the encoded state of the block of a single photon is,  $|\phi_j\rangle = \frac{1}{\sqrt{n}} [|D_1\rangle + \sum_{k=2}^n (-1)^{j'_{k-1}} |D_k\rangle]$ , where  $j'_k = \sum_{l=1}^k j_l$ .

Given the above encoding scheme, we can construct the corresponding states in the entanglement-based protocol. The equivalence between the entanglement-based and the prepare-and-measure protocols is obtained by following the technique by Shor and Preskill [3]. For each encoding block Alice prepares additional  $(n-1)$  qubits which are stored without disturbances in her own quantum memory throughout the protocol. These qubits, labeled with  $A_1, \cdots, A_{n-1}$ , are entangled with a single photon, labeled  $B$  in the corresponding block, which are described as

$$|\phi\rangle = \frac{1}{\sqrt{2^{n-1}}} \sum_{j=0}^{2^{n-1}-1} [(|j_1\rangle_{A_1} \cdots |j_{n-1}\rangle_{A_{n-1}}) \otimes |\phi_j\rangle_B]. \quad (1)$$

On Bob's side, after he receives the single photons, he first applies quantum nondemolition (QND) measurement to determine the number of incoming photons in a block and discard the blocks with multi photons or the vacuum. This step is necessary because the following entanglement purification protocol requires a well-defined qubit on his side. The QND measurement commutes with all other operations on Bob's side. Therefore, in the prepare-and-measure protocol, Bob can replace the QND measurement by the photon number resolving (PNR) detectors which are capable of discriminating the vacuum, the single-photon state, and multiphoton states. Particularly, by using two PNR detectors, Bob only accepts the instances when one detector obtains the single-photon state and the other one obtains the vacuum state in one block.

Then, the single photon goes to a 1-bit delay M-Z interferometer, shown in Fig. 1. An incoming photon state in  $|D_k\rangle$ , is split into four different photon states in the two output ports and two consecutive time slots. Particularly, we obtain  $a_k^\dagger \Rightarrow \frac{1}{2}(u_k^\dagger + iv_k^\dagger + u_{k+1}^\dagger - iv_{k+1}^\dagger)$ , where  $u_k^\dagger$  and  $v_k^\dagger$  are the creation operators of the pulses of the time slot  $k$  in the two output ports. For convenience, Bob applies a  $\pi/2$  phase rotation on each pulse in the bottom output port. Given  $|U_k\rangle = u_k^\dagger|\text{vac}\rangle$  and  $|V_k\rangle = v_k^\dagger|\text{vac}\rangle$ , the operation of the interferometer ( $M_{\text{DPS}}$ ) on each  $|D_k\rangle$  can be written as

$$M_{\text{DPS}}|D_k\rangle = \frac{1}{2}(|U_k\rangle - |V_k\rangle + |U_{k+1}\rangle + |V_{k+1}\rangle). \quad (2)$$

Bob further applies a hypothetical filtering operation in order to project the single photon into a two-level state required for the entanglement purification protocol. The filter operation is described by a set of Kraus operators  $F = \{F_1, F_2, \dots, F_n\}$ , namely,  $F_l = |U_l\rangle\langle U_l| + |V_l\rangle\langle V_l|$ , for  $l = 2, \dots, n$  and  $F_1 = I - \sum_{l=2}^n F_l$  in which  $I$  is the identity matrix, namely,  $I = \sum_{l=1}^{n+1} (|U_l\rangle\langle U_l| + |V_l\rangle\langle V_l|)$ . Note that the projection operators commute with each other and represent monitoring the time slots of the detection events. Bob then publicly announces which time slot the photon was projected to. Alice and Bob will discard the inconclusive blocks where Bob obtains  $F_1$ . By its projection to a certain time slot  $l$ , the photon is in a well-defined qubit

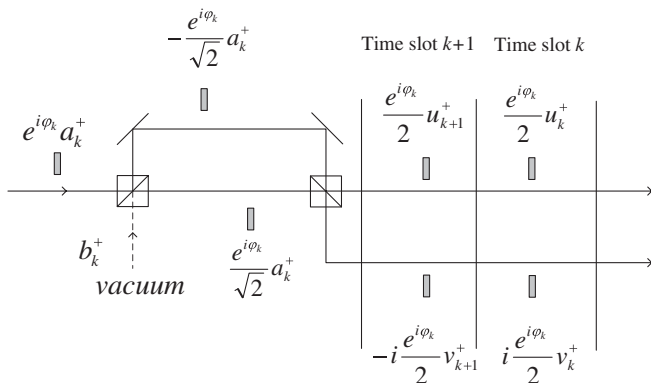


FIG. 1. Schematics of the 1-bit delay M-Z interferometer

state. Particularly, we can define the Z-basis of the photon as  $\{|U_l\rangle, |V_l\rangle\}$ , representing whether it travels along the top or bottom output port of the interferometer. Accordingly, we define Bob's Pauli operators as  $Z_{B_l} = |U_l\rangle\langle U_l| - |V_l\rangle\langle V_l|$  and  $X_{B_l} = |U_l\rangle\langle V_l| + |V_l\rangle\langle U_l|$ .

Finally, Alice and Bob should tackle the eavesdropping and the channel errors. On the one hand, when the channel is ideal and no eavesdropping exists, it is easy to show that Alice and Bob obtain a maximally entangled pair from each projected qubit. Particularly, Alice discards all the qubits on her side with the label other than  $l-1$ , in which  $l$  is Bob's projection outcome. Mathematically, it is equivalent to partially tracing these qubits in the state of Eq. (1). Combining with Bob's filter projection, Alice and Bob share the Bell state, namely,  $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_{A_{l-1}} \otimes |U_l\rangle_B + |1\rangle_{A_{l-1}} \otimes |V_l\rangle_B)$ .

On the other hand, if the channel is noisy or there is an eavesdropper, Alice and Bob share a corrupted two-qubit state. In this case, Alice and Bob employ an appropriate entanglement purification protocol based on Calderbank-Shor-Steane (CSS) code [11], to distill the Bell state. If the entanglement purification protocol succeeds, the resulting smaller set of states shared by Alice and Bob will have very high fidelity. Using the argument that high fidelity implies low entropy [12] or compossibility argument [13], Alice and Bob can generate an unconditionally secure key by measuring the distilled states in their own respective Z-basis. Therefore, the key to the unconditional security proof is whether they can estimate the bit error rate and the phase error rate, which is necessary for choosing an appropriate CSS code for the successful purification. As for the bit errors, Alice and Bob can estimate them by using test bits. However, in the prepare-and-measure protocol, since they do not directly measure the phase errors of the test bits, they have to upper bound them only from the observed quantities.

In what follows, we concentrate only on the untested bits, and for the estimation of the phase error rate, we appeal to Azuma's inequality [14,15]. First, we define  $p_{b,l}^{(k)}$  as the probability of observing a bit error in the  $l$ th time slot of the  $k$ th photon pair. We allow  $p_{b,l}^{(k)}$  to be dependent on the previous  $k-1$  events, in other words, this probability is a conditional probability. Moreover, we define  $Ne_{b,l}$  as the number of the actual bit errors in the  $l$ th time slot after  $N$ -photon-pair emission. Similarly, we can define the sequence  $p_{p,l}$  and  $e_{p,l}$  for the phase errors in the  $l$ th time slot. A consequence of Azuma's inequality states that  $\Pr[|e_{\Lambda,l} - \frac{\sum_{k=1}^N p_{\Lambda,l}^{(k)}}{N}| \geq \epsilon] \leq 2e^{-N\epsilon^2/2}$ , for arbitrary positive number  $\epsilon$ , both  $\Lambda \in \{b, p\}$ , and all conclusive time slots  $l$  [15]. Therefore, if we can find the relation,  $\sum_{l=2}^n p_{p,l}^{(k)} \leq \sum_{l=2}^n C_l p_{b,l}^{(k)}$  for certain  $C_2, \dots, C_n$ , the total phase error rate  $e_p = \sum_{l=2}^n e_{p,l}$  can be bounded by the same relation, namely,  $e_p \leq \sum_{l=2}^n C_l e_{b,l}$ . On the other hand, the random sampling theory states that  $e_{b,l}$  is close

to the measured bit error rate on the test bits with a probability exponentially close to 1 [12].

Eve's most general attacks entangle the whole blocks with her ancilla. Focusing on one certain block, e.g., the  $k$ th block, the attacks can be reduced to a Kraus operator acting only on this block, by the following two steps: First, since Azuma's inequality requires conditional probabilities, we suppose that Alice and Bob have performed fictitious Bell measurements to test errors on the previous  $(k - 1)$  blocks. We project all the previous systems according to the outcomes and trace out the  $(k - 1)$  blocks. Second, we trace out the  $(k + 1)$ th  $\dots$ ,  $n$ th blocks and Eve's ancilla. The resulting operator is  $\Phi_k(\rho_k) = \sum_s E_s^{(k)\dagger} \rho_k E_s^{(k)}$ , where  $\rho_k$  is the unperturbed state of the  $k$ th block. Because the actual measurement outcomes and Eve's coherent attacks are unknown, the operator  $E_s^{(k)}$  is arbitrary and dependent on arbitrary measurement outcomes of the previous  $(k - 1)$  blocks [15].

The linearity of the Kraus operator allows us to consider only one of its components,  $E_s^{(k)} = (a_{ij})$ , an arbitrary  $(n \times n)$ -dimension matrix acting on the single-photon state. The corrupted state of the  $k$ th block becomes  $E_s^{(k)}|\phi\rangle$ . The final state after Bob's interferometer and the filter operation is  $|\phi_l^{(k)}\rangle = F_l M_{\text{DPS}} E_s^{(k)}|\phi\rangle$ . Therefore, for this time slot, we can obtain the possibilities  $p_{b,l}^{(k)} = \langle \phi_l^{(k)} | \frac{1 - Z_{A_{l-1}} Z_{B_l}}{2} | \phi_l^{(k)} \rangle$  and  $p_{p,l}^{(k)} = \langle \phi_l^{(k)} | \frac{1 - X_{A_{l-1}} X_{B_l}}{2} | \phi_l^{(k)} \rangle$ , which are conditioned on arbitrary previous events. The calculations show that

$$p_{b,l}^{(k)} = \frac{1}{4n} [ |a_{l-1,l-1} - a_{l,l}|^2 + |a_{l-1,l} - a_{l,l-1}|^2 + (|a_{l-1,1}|^2 + \dots + |a_{l-1,l-2}|^2 + |a_{l-1,l+1}|^2 + \dots + |a_{l-1,n}|^2) + (|a_{l,1}|^2 + \dots + |a_{l,l-2}|^2 + |a_{l,l+1}|^2 + \dots + |a_{l,n}|^2) ], \quad (3)$$

$$p_{p,l}^{(k)} = \frac{1}{2n} [ |a_{l,1}|^2 + \dots + |a_{l,l-1}|^2 + |a_{l-1,l}|^2 + \dots + |a_{l-1,n}|^2 ]. \quad (4)$$

By observing that for any two complex numbers  $a$  and  $b$ ,  $|a|^2 + |b|^2 \leq \frac{3+\sqrt{5}}{2}(|a-b|^2 + |a|^2)$ , we derive that  $2n \sum_{l=2}^n p_{p,l}^{(k)} - 4n \sum_{l=2}^n p_{b,l}^{(k)} \leq [ (|a_{1,2}|^2 + |a_{2,1}|^2) - (|a_{1,2} - a_{2,1}|^2 + |a_{2,1}|^2) ] + [ (|a_{n-1,n}|^2 + |a_{n,n-1}|^2) - (|a_{n-1,n} - a_{n,n-1}|^2 + |a_{n-1,n}|^2) ] \leq (1 + \sqrt{5})/2 [ (|a_{1,2} - a_{2,1}|^2 + |a_{2,1}|^2) + (|a_{n-1,n} - a_{n,n-1}|^2 + |a_{n-1,n}|^2) ] \leq (1 + \sqrt{5})/2 \times 4n \sum_{l=2}^n p_{b,l}^{(k)}$ , or equivalently  $\sum_{l=2}^n p_{p,l}^{(k)} \leq (3 + \sqrt{5}) \times \sum_{l=2}^n p_{b,l}^{(k)}$ . It should be emphasized that the relations are general for arbitrary matrix component and arbitrary previous measurement outcomes. Using Azuma's inequality, we therefore obtain

$$e_p \leq (3 + \sqrt{5}) \sum_{l=2}^n e_{b,l} = (3 + \sqrt{5}) e_b, \quad (5)$$

where  $e_b = \sum_{l=2}^n e_{b,l}$  is the total bit error rate over all conclusive time slots. The derivation clearly demonstrates the essence of DPS-QKD in which the upper bound of the phase error rate in certain time slot can only be estimated by combining the bit error rates in the same and adjacent time slots. Note that this upper bound does not apply to the  $n = 2$  case in which the phase error rate can be as high as 50% and we have no chance to generate the secret key. This is the case where Alice uses two orthogonal states and Eve has free access to the information. The upper bound is valid for  $n \geq 3$  in which the states are mutually nonorthogonal and no unambiguous state discrimination exists. This result shares some similarity in the security proof of the Bennet 1992 protocol [16].

Combining the above three arguments, we can derive the unconditionally secure key generation rate of the entanglement-based DPS-QKD and the single-photon DPS-QKD, namely,

$$R_{\text{DPS}} \geq p_{\text{DPS}} [1 - H(e_b) - H((3 + \sqrt{5})e_b)], \quad (6)$$

where  $H(x)$  is the binary Shannon entropy, namely,  $H(x) = -x \log_2(x) - (1-x) \log_2(1-x)$ , and  $p_{\text{DPS}}$  is the conclusive detector click rate per pulse in DPS-QKD.

Finally, we compare the key generation rates of the unconditionally secure BB84 protocol ( $R_{\text{BB84}}$ ) [3], DPS-QKD against general attack for individual photons ( $R_{\text{IND}}$ ) [9], and unconditionally secure DPS-QKD ( $R_{\text{DPS}}$ ). We assume single-photon sources in all three protocols. In the presence of channel losses, we express that  $R_{\text{BB84}} = p_{\text{BB84}} [1 - 2H(e_b)]$ , and  $R_{\text{IND}} = p_{\text{IND}} \{ -\log_2 [1 - e_b^2 - \frac{(1-6e_b)^2}{2}] - H(e_b) \}$ , where  $p_{\text{BB84}}$  and  $p_{\text{IND}}$  are the conclusive detector click rates per pulse in the corresponding protocols. Here we adopt the result of DPS-QKD against general attack for individual photons [9] to the case with a single-photon source. We assume that the coding efficiency for the bit error correction approaches to Shannon limit in all three cases. When  $R_{\text{BB84}}$ ,  $R_{\text{IND}}$  and  $R_{\text{DPS}}$  hit zero, the upper bound of the tolerable bit error rates for three protocols are found to be 11%, 6.09%, and 4.12%, respectively. Note that in DPS-QKD the phase error rate is indirectly estimated by  $M_{\text{DPS}}$  and the filter while it is directly estimated in the BB84 protocol. This is an essential insight we obtained in this Letter, and this poor estimation results in lower error rate threshold of DPS-QKD compared to the one of the BB84 protocol.

To simulate the resulting key generation rates, we take the parameters from Ref. [7], where the dark count rate, the time window and the baseline error rate are 50 Hz, 50 ps, and 2.3%, respectively. Therefore, the dark count rate per detector per time slot is  $d = 2.5 \times 10^{-9}$ . We assume that all protocols use two detectors. So  $p_{\text{BB84}} = (\eta + 2d)/2$ , where  $\eta$  is the total efficiency including the channel, the detectors and all other devices. In DPS-QKD, we further assume that the loss event happens equally on every pulse in the transmission, so that the probability of getting a conclusive event is  $(n-1)/n$ . On the other hand, a dark

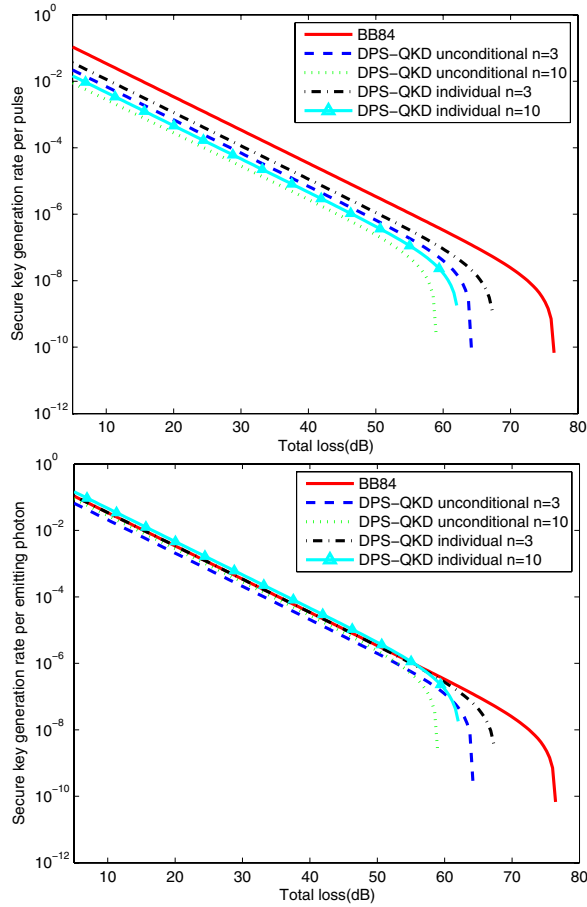


FIG. 2 (color online). Secure key generation rates per pulse (a) and per emitting photon (b) as a function of the total loss. Red solid line:  $R_{\text{BB84}}$ ; blue dashed line and green dash-dot line:  $R_{\text{DPS}}$  with  $n = 3, 10$ , respectively; black dash-dot line and light blue solid line with triangles:  $R_{\text{IND}}$  with  $n = 3, 10$ , respectively.

count can occur in every time slot with equal probability. Therefore, by noting that Bob obtains at most 1 photon out of a block with  $n$  pulses,  $p_{\text{IND}} = p_{\text{DPS}} = \eta(n-1)/n^2 + 2d(n-1)/n$ . Moreover, the bit error rates can be modeled as  $e_b = (e\eta + d)/(\eta + 2d)$  for the BB84 protocol and  $e_b = [e\eta(n-1)/n^2 + d(n-1)/n]/[\eta(n-1)/n^2 + 2d(n-1)/n]$  for DPS-QKD, where  $e$  is the base-line error rate given above.

Figures 2(a) and 2(b) illustrate the secure key generation rates per pulse and the energy efficiencies, namely, the secure key generation rates per emitting photon. As expected, the unconditionally secure key generation rates and the upper bound of the tolerable bit error rate of DPS-QKD are lower than those of DPS-QKD against general attack for individual photons. From Fig. 2(b), larger  $n$  has higher energy efficiency because every photon received by Bob has lower chance to be discarded. However, larger  $n$  will decrease the probability of getting a signal from a pulse and increase the dark count rate per block, and thus lead to lower secure key rate per pulse and lower achievable distance as shown in Fig. 2(a). Based on these observations,

we find that  $n = 3$  yields the optimal secure key generation rate per pulse and the maximum achievable distance.

In conclusion, we have proven the unconditional security of DPS-QKD with a single-photon source and evaluated its secure key generation rate. The security is based on the nonorthogonality of the encoding states for  $n \geq 3$  and Bob's 1-bit delay operation. We hope that our unconditional security proof is a first step toward the security proof of coherent state DPS-QKD.

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