Can Closed Timelike Curves or Nonlinear Quantum Mechanics Improve Quantum State Discrimination or Help Solve Hard Problems?

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We study the power of closed timelike curves (CTCs) and other nonlinear extensions of quantum mechanics for distinguishing nonorthogonal states and speeding up hard computations. If a CTC-assisted computer is presented with a labeled mixture of states to be distinguished—the most natural formulation—we show that the CTC is of no use. The apparent contradiction with recent claims that CTC-assisted computers can perfectly distinguish nonorthogonal states is resolved by noting that CTC-assisted evolution is nonlinear, so the output of such a computer on a mixture of inputs is not a convex combination of its output on the mixture's pure components. Similarly, it is not clear that CTC assistance or nonlinear evolution help solve hard problems if computation is defined as we recommend, as correctly evaluating a function on a labeled mixture of orthogonal inputs.

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Introduction.—Physicists and science fiction writers have long been interested in time travel, wherein a person or object travels backward in time to interact with a younger version of itself. The many studies of such closed timelike curves have led to the general conclusion that, while conditions for their creation may not arise in typical astrophysical or cosmological settings, in principle there seems to be no barrier to their existence [1–5].

In the context of quantum computation, the most widely accepted model of time travel, due to Deutsch [6], involves a unitary interaction U of a causality-respecting (CR) register with a register that traverses a closed timelike curve (CTC). The physical states of Deutsch's theory are the density matrices of quantum mechanics, but the dynamics are augmented from the usual linear evolution. For each initial mixed-state ρ_{CR} of the CR register, the CTC register is postulated to find a fixed-point ρ_{CTC} such that

$$\mathrm{Tr}_{\mathrm{CR}}(U\rho_{\mathrm{CR}} \otimes \rho_{\mathrm{CTC}} U^{\dagger}) = \rho_{\mathrm{CTC}}. \tag{1}$$

The final state of the CR register is then defined in terms of the fixed point as

$$\rho'_{\mathrm{CR}} = \mathrm{Tr}_{\mathrm{CTC}}(U\rho_{\mathrm{CR}} \otimes \rho_{\mathrm{CTC}} U^{\dagger}). \tag{2}$$

The induced mapping $\rho_{\mathrm{CR}} \to \rho'_{\mathrm{CR}}$ is nonlinear because the fixed point ρ_{CTC} depends on the initial state ρ_{CR}. The nonlinear evolution leads to various puzzling consequences considered below, but, because the fixed point is allowed to be a mixed state, it always exists [6], thereby avoiding the notorious "grandfather paradox" wherein some initial conditions lead to no consistent future [7].

In Deutsch's model, the mixed-state fixed-point ρ_{CTC} explicitly begins in a product state with the CR register. Thus, the Universe may evolve from a pure to mixed state, which is not normally allowed by quantum mechanics. To recover a pure state picture, Deutsch appeals to the multiverse of the many-worlds interpretation, where the CTC system in our world is entangled with other worlds' CTC and CR systems. This kind of mixed state runs counter to the "church of the larger Hilbert space" philosophy applicable to CTC-free quantum mechanics, which views mixed states as always being subsystems of larger entangled pure systems in *this* universe.

To illustrate Deutsch's model, consider putting half of a maximally entangled state into a CTC (Fig. 1). There are now two causality-respecting qubits, A and B, and a single

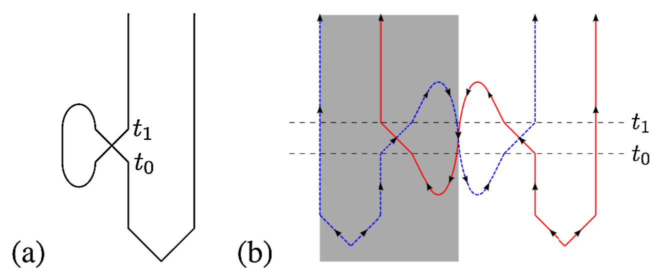

FIG. 1 (color online). Sending half of an EPR pair along a CTC. (a) Single universe picture. An EPR pair, $|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}} \times (|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$, is created in the distant past. At time t_0, a qubit emerges from the CTC and at time t_1, half of the EPR pair is put into the CTC. According to Deutsch's prescription, the density matrix of the CTC system at t_0 is equal to the CTC density matrix at t_1. Nevertheless, the joint state at any time after t_1 is a product state. (b) Multiple universe picture. In both universes, an EPR pair is created in the distant past. At time t_0, a qubit emerges from the CTC in each universe. At time t_1 in each universe, half of an EPR pair is put into the CTC and goes back in time to emerge at t_0 in the other universe. Each EPR particle originally created is entangled with a partner in the other universe and in a product state with the other particle in its own universe.

CTC qubit. The unitary of Eq. (1) is the swap operation between CTC and B. Finding the fixed point gives $\rho_{\mathrm{CTC}} = \frac{1}{2}I$, which along with Eq. (2) gives a final state of $\rho'_{AB} = \frac{1}{4}I_A \otimes I_B$ on the causality-respecting qubits. Strangely, not only does the CTC cause an evolution from a pure to mixed state, but the simple act of sending B along a CTC disentangles it from A. A pure state is recovered by considering both our initial universe and the universe with which the CTC interacts.

Distinguishing states.—Our work is motivated by [10], which explored the benefits of CTCs for state discrimination. There it was shown that for any pair of pure states, $|\phi_0\rangle$ and $|\phi_1\rangle$, there is a CTC-assisted circuit that maps these to orthogonal states $|0\rangle$ and $|1\rangle$, respectively. This was interpreted as distinguishing nonorthogonal states, an impossibility in standard quantum mechanics. It was also shown that the linearly dependent set $\{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}$, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, can be mapped to four orthogonal states using a CTC. Interpreting this as distinguishing these states leads us to a truly remarkable conclusion: a CTC can be used to distinguish $I/2 = \frac{1}{2}|0\rangle \times \langle 0| + \frac{1}{2}|1\rangle\langle 1|$ from $I/2 = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$. Apparently, a CTC lets us distinguish *identical* states. Of course, it is not entirely clear what this means.

The authors of [10] knew something had gone awry, and speculated that either their own analysis or Deutsch's model must be wrong. To resolve this conundrum, we look more closely at what it means to discriminate among quantum states. Discrimination is necessarily *adversarial*, in the sense that a referee, Rob, presents the discriminator, Alice, with a system prepared in an unknown state $|\phi_0\rangle$ or $|\phi_1\rangle$. Before Rob gives her this system, she does not know which state he will prepare, but after some processing, she should be able to tell Rob whether it was $|\phi_0\rangle$ or $|\phi_1\rangle$. Since Rob will choose the state according to some physical process and must remember his choice in order to check that Alice has succeeded, the joint state of Alice and Rob before any distinguishing operation is

$$\rho_{RA} = \sum_{x=0}^{1} p_x |x\rangle\langle x|_R \otimes |\phi_x\rangle\langle\phi_x|_A. \tag{3}$$

Alice will now apply some operation to the A system. We will say she has succeeded if the joint state afterwards is

$$\rho'_{RA} = \sum_{x=0}^{1} p_x |x\rangle\langle x|_R \otimes |x\rangle\langle x|_A. \tag{4}$$

Our formulation of the problem may seem obvious, and even a bit pedantic, but as we will now see, it has major consequences for the power of CTCs: they are entirely useless for state discrimination. To see this, suppose we have a CTC-assisted protocol for distinguishing $|\phi_0\rangle$ and $|\phi_1\rangle$ that takes a causality-respecting input A and closed timelike curve register CTC. The causality-respecting region consists of R and A, with the fact that Alice does not have access to R reflected in the restriction of Eq. (1) to $U = I_R \otimes V_{A,\mathrm{CTC}}$. Even without access to a CTC, because she knows p_x and $|\phi_x\rangle$ (though not the particular value of x), she can solve the fixed-point problem (1) to get ρ_{CTC}. So, she can prepare a quantum state ρ_{CTC} and, given a state to distinguish on A, apply V to the joint $ACTC$ system and generate the same output state ρ'_{RA} as if she actually had a CTC. In short, Alice can simulate the help of a CTC by solving the fixed-point problem herself, eliminating any advantage the CTC may have offered.

How do we reconcile the fact that CTCs do not improve state discrimination with the finding of [10] that any pair of pure states can be mapped to orthogonal outputs using a CTC? We must be careful to avoid falling into the following "linearity trap": while in standard quantum mechanics the evolution of a mixture is equal to the corresponding mixture of the evolutions of the individual states, in a nonlinear theory, this is not generally true (see Fig. 2). Thus, while the circuit of [10] (see Fig. 3) can map $|x\rangle_R|\phi_x\rangle_A \to |x\rangle_R|x\rangle_A$, it does not map the mixed-state Eq. (3) to the desired output (4) but rather to

$$\left(\sum_{x=0}^{1} p_x |x\rangle\langle x|_R\right) \otimes \rho'_A. \tag{5}$$

The output ρ'_A depends on the ensemble $\{p_x, |\phi_x\rangle\}$ but not on the particular value of x. Indeed, even when presented with a superposition of states, $\sum_x \sqrt{p_x}|x\rangle_R|\phi_x\rangle_A$, the circuit fails. The correlations between R and A are completely broken, reflecting the disentangling nature of Deutsch's model of CTCs.

Computational consequences.—We now focus on the computational power of closed timelike curves. Several authors have concluded that access to CTCs would have substantial computational benefits. For example, [11] suggested that a CTC would allow a classical computer to efficiently factor composite numbers and gave hints that a CTC-enhanced computer may be much stronger. In [12], it

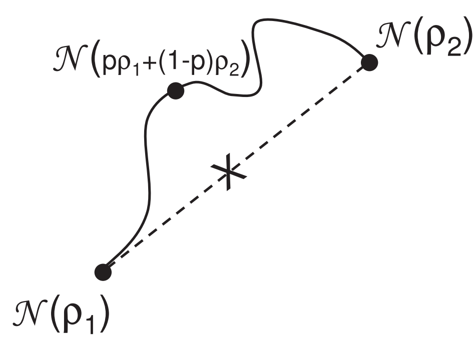

FIG. 2. The linearity trap. The action of a nonlinear map $\mathcal{N}$ on states ρ_1 and ρ_2 does not determine the action on their mixture. An example of such a map is the evolution of states in the CTC model. So, although a CTC allows nonorthogonal pure states to be mapped to orthogonal outputs, this does not suffice to identify the states in an unknown mixture. Similarly, the apparent power of CTC-assisted computations is not enough to allow a user to sample the correct output of the computation over an arbitrary distribution of inputs.

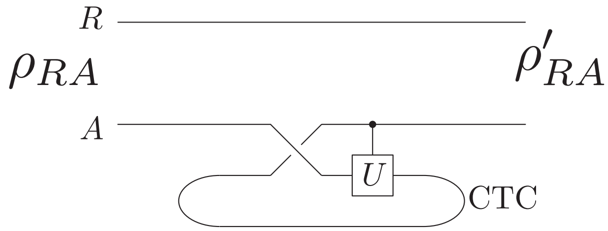

FIG. 3. The state discrimination circuit of [10]. The circuit on A and CTC is designed to distinguish pure states $|0\rangle$ and $|\psi\rangle$. U is chosen with $U|\psi\rangle = |1\rangle$ which leads to fixed points $|0\rangle\langle 0|$ when $|0\rangle$ is input and $|1\rangle\langle 1|$ when $|\psi\rangle$ is input. However, faced with the task of distinguishing an unknown mixture labeled by R as in Eq. (3), the output $\rho'_{RA} = \rho_R \otimes \rho'_A$. The output of the circuit is independent of the identity of the state.

was argued that a CTC-assisted quantum computer could efficiently solve NP-complete problems, a feat widely believed impossible for a quantum computer alone. The strongest results about computation using CTCs are those of [13], where it was reported that the power of a polynomial time bounded computer (either classical or quantum) assisted by CTCs is exactly PSPACE, the class of problems that can be solved in a space polynomial in the problem size but potentially exponential time. Because PSPACE is thought to contain many problems that cannot be solved efficiently without CTC assistance, this would suggest that CTCs are extremely useful for computation.

Analyzing the power of CTCs is a subtle business, as we saw with state discrimination. To understand what is going on it is useful to spell out exactly what we mean by "computing." We first have to ask how the input to the calculation is chosen. If it is chosen by some physical process, the inputs have some probability distribution that depends on the selection procedure. As we have seen, for a nonlinear theory, the performance of a circuit depends on the probability distribution over the inputs. So, probably the strongest form of computation would be to provide the correct answer for every input distribution.

In [13] (and implicitly in [12]), the class $\mathrm{BQP}_{\mathrm{CTC}}$ is defined, where BQP stands for bounded error quantum polynomial time and the subscript refers to its augmentation by a CTC. By their definition, an algorithm succeeds if it gives the correct answer on every pure state input. In fact, in all previous work on computation with CTCs it is shown that for a fixed pure state input (and even for *all* pure state inputs), the proposed circuit reaches the correct output. However, to argue that it follows that a physical computer would work on every input distribution would be to fall prey to the linearity trap.

It is easy to check that the circuits of [12,13] for computing a function $F(x)$, when applied to a uniform mixture of inputs (with an external referee remembering which one has been supplied), do not generate the state

$$\frac{1}{X}\sum_{x=1}^{X} |x\rangle\langle x|_R \otimes |F(x)\rangle\langle F(x)|_A$$

but give a product state similar to Eq. (5). The output of the circuit is uncorrelated with the input. Thus, we believe claims that a quantum computer with CTC assistance can efficiently solve NP-complete and PSPACE-complete problems are dubious, at least for the natural definition of computation as the ability to find the correct output no matter how the input is chosen.

Given their definition of $\mathrm{BQP}_{\mathrm{CTC}}$, the arguments in [12,13] are valid, but this definition is problematic because it implicitly limits the computer to operating on a single input rather than a range of possible inputs. The physical interpretation of a single input might be that one has made a firm and unwavering decision to use a CTC to solve a particular problem (e.g., whether black has a winning strategy in Go), rather than a class of problems, as is usual in computational complexity theory. This decision may as well be taken to have existed since the beginning of time, and cannot depend on any other part of the universe. Only then will the CTC-assisted computer give the desired result. There is no physical problem with this, as it is equivalent to the universe having been created with special objects containing answers to particular questions, but it is not very appealing in terms of the common meaning of computation. For example, one might be disappointed by a Go computer claiming to know the winner of the standard 19×19 game but unable to shed any light on variants using boards of other sizes.

Thus, we suggest a new complexity class $\mathrm{BQPP}_{\mathrm{CTC}}$, whose definition is identical to that of $\mathrm{BQP}_{\mathrm{CTC}}$ of [13], except that the computer must produce correctly correlated mixtures of input-output pairs for all labeled input distributions (and the input is supplied as a string rather than a circuit). We do not know whether $\mathrm{BQPP}_{\mathrm{CTC}}$ is stronger than the unassisted BQP. Since the CTC fixed point is uncorrelated with the inputs to a circuit, it seems like a fairly weak resource, akin to "quantum advice" [14,15]. Fortunately, the argument in [13] that $\mathrm{BQP}_{\mathrm{CTC}} \subset \mathrm{PSPACE}$ holds for our definition of computing, so at least we know that $\mathrm{BQPP}_{\mathrm{CTC}}$ is in PSPACE.

Similar arguments apply to classical complexity classes like P and BPP in the presence of CTCs. If computation is defined in the natural manner we recommend, CTCs have not been shown to enlarge any of these classes.

General nonlinear theories.—Weinberg has proposed a general approach for adding nonlinearities to quantum mechanics [16,17]. It was argued almost immediately that the theory has pathological properties. Notably, [18,19] suggested that the theory gives faster than light communication. Moreover, modification to eliminate this problem gives communication between branches of the wave function, dubbed the "Everett Phone" [18]. It was also argued [20] that it violates the second law of thermodynamics. Finally, [21] argued that *any* nonlinear version of quantum mechanics allows the efficient solution of NP-complete and #P-complete problems.

In the follow-up work to [16,17]—the instantaneous communication of [18,19], the second law violation of

[20], and the computational speed-up of [21]—the arguments proceed by considering the evolution of some pure state, then inferring the induced evolution of their mixture. It is the linearity trap again. For example, just as the CTC-circuits of [12,13] fail on a mixed state, the circuits of [21] using the nonlinearities of [16–18] give outputs that are uncorrelated with their inputs when applied to a labeled mixture, resulting in no computational speed-up. Because the linearity trap is so enticing, we propose a rule of thumb for dealing with nonlinear theories:

The Principle of Universal Inclusion: The evolution of a nonlinearly evolving system may depend on parts of the universe with which it does not interact.

This principle reflects the fact that (1) calculations ignoring any part of the universe invite the linearity trap, and (2) theories formulated only on subsystems are incomplete. The parts of the universe that are perilous to ignore in the nonlinear theories above are the systems used to select inputs to computational or information theoretic problems. In linear quantum mechanics, this causes no problems because for an evolution $\mathcal{N}$, we have

$$I \otimes \mathcal{N}\left(\sum_i p_i |i\rangle\langle i| \otimes \phi_i\right) = \sum_i p_i |i\rangle\langle i| \otimes \mathcal{N}(\phi_i) \qquad (6)$$

but in other theories this is not so. Perhaps this is what Polchinski [18] was driving at in his discussions of the "Everett phone," cautioning against "treat[ing] macroscopic systems as though they begin in definite macroscopic states" instead of considering their entire histories.

Discussion.—Much of the apparent power of CTCs and nonlinear quantum mechanics comes from analyzing the evolution of pure states, and extending these results linearly to find the evolution of mixed states. However, because mixed states do not have unique decompositions into pure states, this does not give an unambiguous rule for evolution. Indeed, the very nature and meaning of mixed states may be ill defined in such theories. One could potentially resolve this problem by including additional degrees of freedom identifying the "correct" decomposition of mixed states, which would restore the power of CTCs. Unfortunately, this resolution does not reduce to standard quantum mechanics far from any CTC. We find it more rewarding to concentrate on theories that do, such as Deutsch's formalism. In such theories, we can, far from the CTCs, unambiguously define initial and final mixed states for the tasks of state discrimination and computation. We then find that CTCs do not seem to help much in their accomplishment.

Besides [6,16–18], there are several models for CTCs and nonlinear quantum mechanics [8,22–26]. Their information processing power is not known, and our work underscores the necessity of clear and well-motivated definitions of the tasks under consideration in any such study.

The reported pathological behavior of nonlinear quantum mechanics could have been construed as explaining why nature chose standard linear quantum mechanics. Our findings that many of these behaviors do not survive careful scrutiny suggest that a well-behaved nonlinear theory may be possible. In fact, as pointed out in [18], we could in principle have large nonlinearities in a global theory that have little or no consequence for experiments on small systems. It would have been nice to rule out nonlinearity by causality or a prohibition on computational extravagance, but it seems that we cannot.

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