Universal Probability Distribution Function for Bursty Transport in Plasma Turbulence

I. Sandberg,[1,2] S. Benkadda,[3] X. Garbet,[4] G. Ropokis,[2] K. Hizanidis,[1] and D. del-Castillo-Negrete[5]

[1]*National Technical University of Athens, Association Euratom—Hellenic Republic, Athens 15773, Greece*
[2]*National Observatory of Athens, Institute for Space Applications and Remote Sensing, Penteli 15236, Greece*
[3]*France-Japan Magnetic Fusion Laboratory, LIA 336/UMR 6633 CNRS-Université de Provence, Marseille, France*
[4]*I.R.F.M., CEA Cadarache, St Paul-Les-Durance 13108, France*
[5]*Oak Ridge National Laboratory, Oak Ridge, Tennessee 37831-8071, USA*

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Bursty transport phenomena associated with convective motion present universal statistical characteristics among different physical systems. In this Letter, a stochastic univariate model and the associated probability distribution function for the description of bursty transport in plasma turbulence is presented. The proposed stochastic process recovers the universal distribution of density fluctuations observed in plasma edge of several magnetic confinement devices and the remarkable scaling between their skewness S and kurtosis K. Similar statistical characteristics of variabilities have been also observed in other physical systems that are characterized by convection such as the x-ray fluctuations emitted by the Cygnus X-1 accretion disc plasmas and the sea surface temperature fluctuations.

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Plasma turbulence and the associated heat and particle transport play a major role in the levels of plasma confinement in magnetic fusion devices. In the plasma edge the turbulent fluctuations are large and bursty due to the convective motion of strongly nonlinear structures formed during the nonlinear saturation of plasma instabilities. Such coherent structures in an unambiguous manner effectively contribute to radial transport and to intermittency [1]. As a result, the transport process departs from the diffusive picture associated with the Gaussian case of weak independent fluctuations.

In order to understand the underlying mechanism of the turbulent transport in the plasma edge, it is crucial to investigate the statistical characteristics of these bursty fluctuations. Experimental investigations have indeed revealed the bursty nature of particle transport in the scrape of layer (SOL) of magnetically confined plasmas [2]. The appearance of structures such as plasma blobs, avaloids [3] is attributed to the formation of field-aligned structures—induced by the charge separation of the magnetic curvature drifts—that propagate radially far into the SOL.

The statistical behavior of the density fluctuations associated with such bursty dynamics has been investigated in several experiments. A comprehensive study [4], that included measurements from various magnetic confinement devices—including TORE SUPRA, MAST and ALCATOR C-MOD—showed that the associated extreme probability distribution functions (PDFs) are universal (see Fig. 1) in the sense that have the same properties in many confinement devices with different configurations.

Remarkably, similar [5] extreme distributions have been also observed for the bursty X-ray fluctuations associated with transport events in the Cygnus X-1 accretion disc plasmas that are linked to instabilities which give rise to turbulent transport and extreme statistics (see Ref. [6] and Fig. 12 therein). However, known forms of extreme PDFs—such as the Fréchet or the Gumbel distributions—do not have [6] the proper form to fit well the experimentally observed distributions associated with bursty convective transport processes.

Bursty dynamics is characterized by strongly non-Gaussian PDFs and nonvanishing probabilities of extreme events. In such dynamical systems, the higher order moments are commonly used to determine the scaling properties of the fluctuating fields. The non-Gaussian features

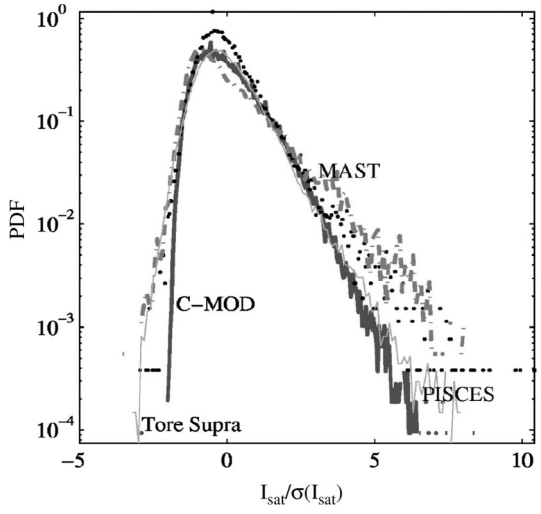

FIG. 1. The PDF plot of the ion saturation current in the Tore Supra (solid line), Alcator C-Mod (thick solid line), MAST (dashed-dotted line), and PISCES (dots). The ion saturation current was normalized to the standard deviation and the integral of the four PDFs is set equal to 1. Figure reprinted with permission from Ref. [4] (http://link.aip.org/link/?PHPAEN/10/419/1), Fig. 3, Copyright 2003, American Institute of Physics.

are usually quantified in terms of skewness and kurtosis of the PDF of fluctuating fields. For a centered random variable $\tilde{x}$, i.e. $\langle \tilde{x} \rangle = 0$, with variance σ^2, skewness is defined by the $S \equiv \langle \tilde{x}^3 \rangle / \sigma^3$ and the kurtosis (in the newer literature) by $K \equiv \langle \tilde{x}^4 \rangle / \sigma^4 - 3$. Skewness is a measure of asymmetry of a PDF; if the left tail is more pronounced than the right tail, the PDF has negative skewness and when the reverse is true, it has positive skewness. Kurtosis measures the excess probability (flatness) in the tails, where excess is defined in relation to a Gaussian distribution.

Using ten thousand observed density fluctuation signals measured in TORPEX, Labit *et al.* [7] showed that a unique parabolic scaling relation holds, $K = 1.502S^2 - 0.226$, between the skewness S and the kurtosis K (see Fig. 1 of Ref. [7]). It was also shown that the PDFs of the measured signals, including those characterized by a negative skewness can be described by a special case of the Beta distribution. The density fluctuations were associated with regimes of drift-interchange (D-I) turbulence generated in regions of bad magnetic field curvature and convected away by the $E \times B$ fluid motion.

Remarkably, there is a striking similarity of the observed $K - S$ scaling with that of sea surface temperature (SST) fluctuations that are governed by advection through ocean currents [8]. Sura and Sardeshmukh [8] proposed a nonlinear Langevin model that can predict the observed scaling in some limits of the model parameters while Krommes [9] generalized it to include linear waves—an essential feature of D-I turbulence—and he numerically calculated the associated PDF that includes four independent parameters. The fact that the same scaling applies to different physical situations leads to conjecturing that the scaling arises due to basic constraints. The general mathematical constraints on the $K - S$ relation, as arise from the definition of kurtosis and skewness [10], do not provide any insight about the observed scaling. A recent comprehensive study on the observed $K - S$ scaling between various fusion devices showed that the data align along parabolic curves [11]. A phenomenological model using the assumption that the fluctuating signals include a linear combination of two basis (Gamma or Beta) PDFs attempting to accommodate experimental evidence was also provided. However, the large number of the free parameters entering to the model and the small difference between using one or a sum of two Beta PDFs make the model less flexible [11].

It becomes evident that these observations among different physical systems reveal a universal character associated with strongly non-Gaussian processes. However, the key question is what kind of underlying mechanism is responsible for the universally observed statistical features and what PDF can describe them. In this Letter, a univariate model for the statistical description of bursty fluctuations is presented, using as an example the aforementioned plasma edge density fluctuations. The associated PDF is derived and it is shown that it recovers the universally observed distributions, documented in Ref. [4] and the remarkable parabolic scaling between skewness and kurtosis as well, documented in Ref. [7]. The derived results have universal character, and thus may be applicable to all aforementioned physical systems.

The nonlinear processes described by the standard models of turbulence in magnetized plasma are quadratic and are linked with small or large scale convection processes attributed to electric drifts. Thus, it is natural one to assume that the universally observed statistical characteristics associated with the bursty behavior of fluctuations may be attributed to processes that emerge from the nonlinear quadratic interaction between turbulent fields. However, extreme statistical features are expected to appear when strong non-Gaussian processes coexist with Gaussian ones. In order to describe the associated universal statistical properties, we propose a univariate non-Gaussian process $W(t)$ given by:

$$W(t) = \frac{Z(t) - \langle Z(t) \rangle}{\sigma_0} + \gamma \frac{Z^2(t) - \langle Z^2(t) \rangle}{\sigma_0^2} \qquad (1)$$

which results from the superposition of a Gaussian $Z(t)$ (with standard deviation σ_0) with a non-Gaussian process attributed to the square of $Z(t)$. The latter corresponds to the strongest quadratic nonlinearity that may arise due to the interaction of fluctuating fields. The coefficient γ, in front of the quadratic non-Gaussian component is a parameter that measures the deviation of $W(t)$ from Gaussianity. The process $W(t)$ may well describe extreme bursty behavior of fluctuations that is characterized by nonlinear structures (blobs, avaloids) that travel (convect) through a sea of Gaussian fluctuations. Figure 2 presents some typical time series $W(t)$ calculated from Eq. (1) by using a centered random Gaussian fluctuation $Z(t)$. In all cases the bursty nature of $W(t)$ is evident. For the sake of simplicity, from now on, we drop the dependence of Z and

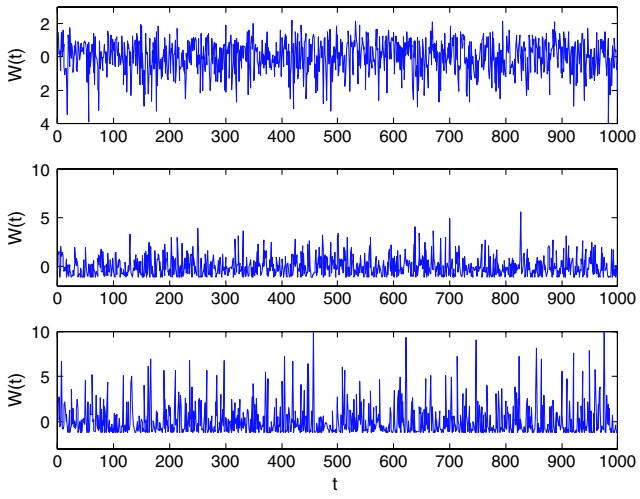

FIG. 2 (color online). (a) Example of bursty time series $W(t)$ for different values of γ: (a) -0.11, (b) 0.031, (c) 0.98, corresponding to negative (top), positive (middle), and significantly large positive (bottom) skewed distributions, respectively.

W on time and consider that Z is a centered Gaussian process. It should be noted here that the process described by Eq. (1) has been presented in the literature [12] as a characteristic simple-to-construct non-Gaussian process and was used as an example for the calculation of the systematic errors of covariances and moments up to fourth order of non-Gaussian time series.

For the derivation of the PDF $p(w)$ of the random process $W(t)$, we express the cumulative distribution function (CDF) $P(w)$ as follows

$$P(w) = \Pr(W \leq w) = \Pr(\gamma(Z - z_1)(Z - z_2) \leq 0), \quad (2)$$

where z_1 and z_2 are the roots of the polynomial

$$f(Z) = \gamma \frac{Z^2 - \sigma_0^2}{\sigma_0^2} + \frac{Z}{\sigma_0} - w. \quad (3)$$

Equation (2) can be expressed in terms of the CDF $P(Z)$ at $z = z_1, z_2$ and $p(w)$ can be derived by a simple differentiation that leads to the following expression:

$$p(w) = \begin{cases} 0, & \text{when } \lambda_w \equiv 4\gamma(w - w_0) < 0 \text{ else;} \\ \sqrt{\frac{2}{\pi\lambda_w}} \cosh\left(\frac{\sqrt{\lambda_w}}{4\gamma^2}\right) \exp\left(-\frac{1+\lambda_w}{8\gamma^2}\right), \end{cases} \quad (4)$$

where $w_0 = -(4\gamma)^{-1} - \gamma$. The value and the sign of parameter γ controls the shape and the range of nonzero values of $p(w)$. For $\gamma = 0$, the $p(w)$ reduces to that of a centered Gaussian random process. For $\gamma > 0$ ($\gamma < 0$) a cut off value w_0 exists and $p(w)$ gets asymmetric presenting long tails in the positive (negative) axis (cf. Fig. 3). The minimum (in absolute sense) value of the cutoff is equal to $w_{0m} = \pm 1$ and corresponds to $\gamma = \mp 1/2$.

In Fig. 4, we have plotted $p(w)$ choosing positive values of γ that range up to $\gamma = 0.2$. The resulting distributions exhibit the same behavior with the measured distributions of plasma edge density fluctuations, as have been observed in several magnetic confinement devices and presented here in Fig. 1. Note, that the characteristic clustering of the PDF curves around 0.05 for values around 2.5 which is experimentally observed (cf. Fig. 1) is recovered by the distribution $p(w)$ (cf. Figs. 3 and 4)—independently on the value of $\gamma > 0$—a characteristic that is not recovered in the plots of PDF in Ref. [9]. Similar features appear also in the PDF of the x-ray observations associated with anomalous transport in accretion disks (cf. Fig. 8 of Ref. [6]).

Unlike, to the BHP distribution [13] which does not have any free parameter, $p(w)$ depends on γ such that its higher order moments can receive multiple values. Furthermore, the existence of a single free parameters allows $p(w)$ to be used for the fitting of experimentally observed distributions. The high order moments of multivariate Gaussian processes can be determined by a simple method based on the Wicks theorem. All odd moments are zero and all even moments can be reduced to homogeneous polynomials. For the considered process W, the values of skewness and kurtosis depend on the value of γ and are given by [12],

$$S = 2\gamma \frac{3 + 4\gamma^2}{(1 + 2\gamma^2)^{3/2}} \quad \text{and} \quad K = 48\gamma^4 \frac{1 + \gamma^2}{(1 + 2\gamma^2)^2}. \quad (5)$$

For $\gamma = 0$, W gets Gaussian and $K = S = 0$. As $|\gamma|$ increases K and S converge to their extremes $K_{\max} = 12$ and $S_{\max} = \pm 2^{3/2}$, respectively, in agreement with the range of the values reported in the Letter by Garcia *et al* [14], where intermittent transport in plasma edge – SOL is numerically investigated. Using the parabolic ansatz, $K = aS^2 + b$ for the parametric relations in Eq. (5), it can be easily found that

$$a(\gamma) = 12 \frac{(1 + \gamma^2)(1 + 2\gamma^2)}{(3 + 4\gamma^2)^2} \text{ and } b = 0. \quad (6)$$

The function $a(\gamma)$ converges rapidly to $a = 3/2$, which is exactly the value of the universally observed parabolic scaling, reported in Refs. [7–9]. It is interesting to compare

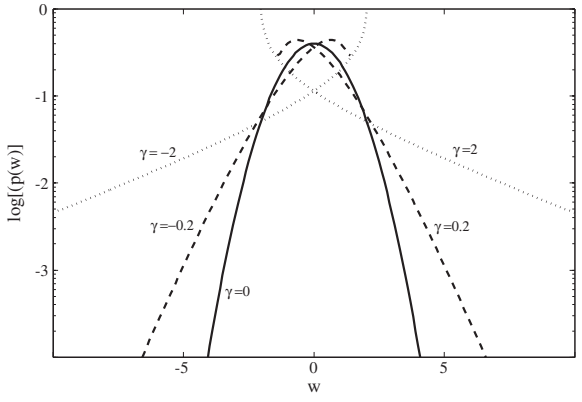

FIG. 3. The family of distributions $p(w)$, for different values of $\gamma = -2, -0.02, 0, 0.02$, and 2. The symmetry with respect to γ is evident. As γ increases the absolute value of the cutoff decreases (increases) for $\gamma < 0.5$ ($\gamma > 0.5$).

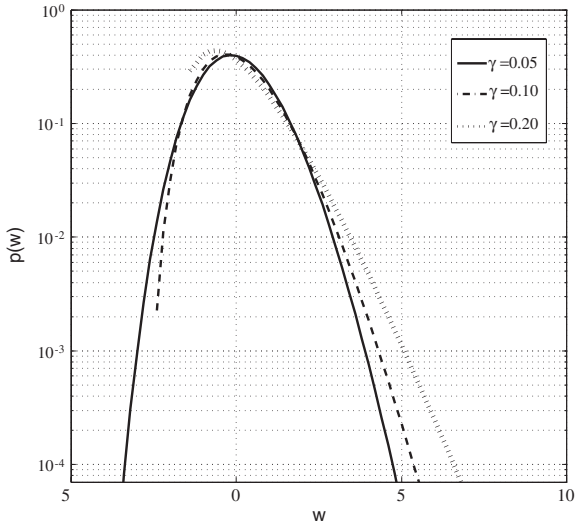

FIG. 4. The family of distributions $p(w)$, for $\gamma = 0.05$, 0.1, and 0.2. The resulting PDFs recover the universally observed distributions presented at Fig. 1.

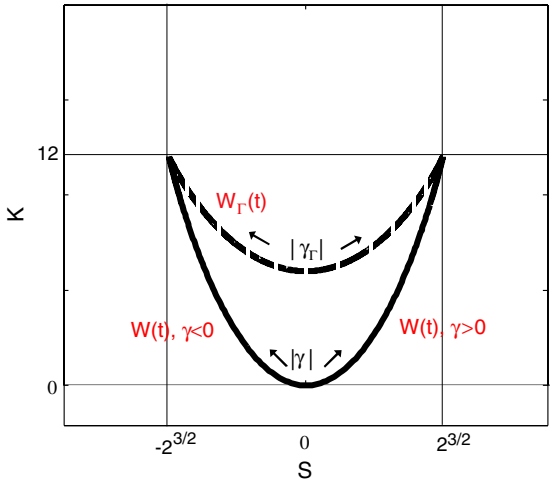

FIG. 5 (color online). The $K - S$ curve for stochastic processes described by quadratic polynomial of Gaussians. The solid line corresponds to $W(t)$ and the dashed line to the $W_\Gamma(t)$ process.

the $K - S$ curve of Eq. (5) to that associated to the product of two central Gaussian processes with correlation γ_Γ, i.e., $W_\Gamma = \gamma_\Gamma Z_1 Z_2$, given by

$$S_\Gamma = -2\gamma_\Gamma \frac{3 + \gamma_\Gamma^2}{(1 + \gamma_\Gamma^2)^{3/2}} \quad \text{and} \quad K_\Gamma = 6\frac{1 + 6\gamma_\Gamma^2 + \gamma_\Gamma^4}{(1 + \gamma_\Gamma^2)^2}, \tag{7}$$

respectively. The parabolic relation, $K = a_\Gamma S^2 + b_\Gamma$, between S_Γ and K_Γ, results to the derivation of the following coefficients:

$$a_\Gamma = \frac{6(1 + \gamma_\Gamma^2)}{(3 + \gamma_\Gamma^2)^2} \quad \text{and} \quad b_\Gamma = 6. \tag{8}$$

Note that for $|\gamma_\Gamma| = 1$, the values of kurtosis and skewness are equal to $K_{\max}$ and $S_{\max}$, while for $\gamma_\Gamma = 0$, are equal to 6 and 0.

Equations (5) and (7) define a closed curve in the $K - S$ space for continuous values of the parameters γ and γ_Γ (see Fig. 5). The low boundary of the curve corresponds to extreme bursty processes W given by Eq. (1) and distributed according to $p(w)$, while the upper boundary corresponds to W_Γ which is distributed according to the "local flux PDF" [see Eq. (7) in Ref. [15]]. The clustering of the $K - S$ values around the $K = 1.5S^2$ curve is clearly seen in Fig. 1 of Ref. [7] and in Fig. 3 of Ref. [8].

It is straightforward to show numerically, that for non-Gaussian processes (described by second order polynomials of Gaussians) the "cloud" of $K - S$ points fall into (not shown here) the closed curve. The latter follow a parabolic trend—similar to the experiments [11]—with scaling that depends on the selection of the polynomial coefficients. Furthermore, for processes described by higher order Gaussian polynomials the "cloud" follows also parabolic trend including much higher values of K and S.

In conclusion, a univariate stochastic model for the description of extreme bursty fluctuations W based on generic properties of quadratic nonlinearities is presented. The model and the associated probability distribution function describe statistical properties of the SOL density fluctuations which are governed by the $E \times B$ convection. The universal character of the stochastic process stems from the fact that the associated extreme PDF recovers properties of variabilities (density, sea temperature, x-ray intensity) observed at boundary regions (SOL, sea surface, accretion edge) of different physical processes that are characterized by convection (electric drift, ocean current, rotation). The proposed univariate stochastic model describes the statistical characteristics of these relaxation phenomena at a state of extreme statistical behavior. It is evident that the parabolic relation between S and K when observed provide relevant information about the underlying processes.

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