Flat 3-brane with Tension in Cascading Gravity

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In the cascading gravity brane-world scenario, our 3-brane lies within a succession of lower-codimension branes, each with their own induced gravity term, embedded into each other in a higher-dimensional space-time. In the $(6 + 1)$-dimensional version of this scenario, we show that a 3-brane with tension remains flat, at least for sufficiently small tension that the weak-field approximation is valid. The bulk solution is singular nowhere and remains in the perturbative regime everywhere.

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An old idea to address the vexing problem of the cosmological constant is to confine the visible Universe on a 3-brane in a higher-dimensional space-time: vacuum energy on the brane curves the extra dimensions, but leaves the $4d$ geometry flat [1]. While tantalizing, this proposal fails as soon as the extra dimensions are compactified; since $4d$ general relativity is recovered below the compactification scale, the theory unavoidably succumbs to Weinberg's no-go theorem [2]. (An alternative strategy is to use compact extra dimensions to suppress radiative corrections to the cosmological constant [3].)

The situation is drastically different, and far more promising, if the extra dimensions have infinite volume [4]. In this case, gravity is approximately $4d$ only at short distances, thanks to an Einstein-Hilbert term on the brane, but becomes *higher dimensional* in the infrared. In the Dvali-Gabadadze-Porrati (DGP) scenario [5] with one extra dimension, the gravitational force law on the brane scales as the usual $1/r^2$ at short distances, but the asymptotes scales as $1/r^3$ at large distances. Gravity therefore behaves as a high-pass filter [6]. This weakening of gravity suggests that vacuum energy, by virtue of being the longest-wavelength source, only *appears* small because it is *degravitated* [6,7].

The degravitation phenomenon is not realized in the original DGP model because the weakening of the force law is too shallow in the infrared [7]. This motivates exploring higher-codimension branes, i.e., a higher-dimensional bulk. Realizing these higher-codimension scenarios has proven difficult. To begin with, the simplest constructions are plagued by ghost instabilities [8,9]. Second, the $4d$ propagator is divergent and must be regularized [10]. Furthermore, for a static bulk, the geometry for codimension $N > 2$ has a naked singularity at finite distance from the brane, for arbitrarily small tension [4]. (Allowing the brane to inflate gives a Hubble rate on the brane *inversely* proportional to the brane tension for codimension $N > 2$ [4].)

Recently, it was argued that these pathologies are resolved by embedding our 3-brane within a succession of higher-dimensional branes, each with their own induced gravity term [11–13]. We refer to this framework as cascading gravity. The induced graviton kinetic term acts as a regulator for the 3-brane propagator [11,12]. In the case $N = 2$ studied in [11], consisting of a 3-brane embedded in a 4-brane within a $(5 + 1)$-dimensional bulk, the ghost is cured by including a sufficiently large tension on the (flat) 3-brane [11,14]. Alternatively, the ghost is also cured when considering a higher-dimensional Einstein-Hilbert term localized on the brane [9,12].

Already with $N = 2$, the solution exhibits degravitation: a 3-brane with tension creates a deficit angle in the bulk while remaining flat [14]. We stress that this self-tuning mechanism crucially relies on the extra dimensions having infinite volume: if the dimensions were compact, the brane tension would have to be tuned against other branes and/or bulk fluxes [15].

Since the deficit angle must be less than 2π, the tension allowed by the solutions considered in [11,14] is bounded by M_6^4, where the $6d$ Planck mass M_6 is itself constrained to be at most $\sim$meV. Given its geometrical nature, this bound is most likely an artifact of the codimension-2 case and is expected to be absent in the higher codimension.

Motivated by these considerations, in this Letter we explore cascading gravity with $N = 3$, consisting of a 3-brane living on a 4-brane, itself embedded in a 5-brane, together in a $(6 + 1)$-dimensional bulk, as sketched in Fig. 1. Including tension on the 3-brane, we derive a solution for which (i) the bulk metric is nonsingular everywhere (except, of course, for a delta-function in curvature at the 3-brane location) and asymptotically flat, and (ii) the induced 3-brane geometry is exactly flat.

Since the metric depends on 3 spatial coordinates, to proceed analytically we restrict ourselves to the weak-field approximation, corresponding to sufficiently small tension. For consistency, we check that our solution remains perturbative everywhere. We are currently working on numerically extending these solutions to the nonlinear regime of large tension.

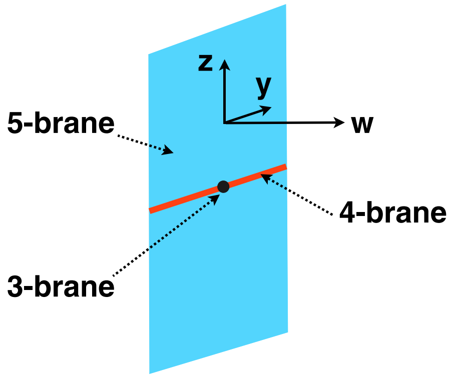

FIG. 1 (color online). Sketch of the codimension-3 cascading gravity setup.

Unlike the case of a pure codimension-3 DGP brane in (6 + 1) dimensions, where the static bulk geometry has a naked singularity for arbitrarily small tension [4], here the bulk metric is completely smooth. This traces back to the cascading mechanism of regulating the propagator: the presence of parent branes removes the power-law divergence in the 4d propagator.

As illustrated in Fig. 1, the 3 extra spatial dimensions are denoted by y, z, and w, with the codimension-1 brane located at $w = 0$, the codimension-2 brane at $z = w = 0$, and the codimension-3 brane at $y = z = w = 0$. Indices in 7d are denoted by $A, B, \ldots$, indices in 6d by $a, b, \ldots$, indices in 5d by $\alpha, \beta, \ldots$, and finally indices in 4d by μ, $\nu, \ldots$

I. Scalar Green's Functions.—In solving for the metric perturbations, it is useful to first consider the scalar Green's functions, determined from the action

$$S = \frac{1}{2}\int d^7x\Psi[M_7^5\Box_7 + \delta(w)M_6^4\Box_6 + \delta^2(z,w)M_5^3\Box_5 + \delta^3(y,z,w)M_4^2\Box_4]\Psi, \tag{1}$$

where M_d denotes the "Planck" mass in d dimensions. The model has three cross-over scales:

$$m_5 = \frac{M_5^3}{M_4^2}, \qquad m_6 = \frac{M_6^4}{M_5^3}, \quad \text{and} \quad m_7 = \frac{M_7^5}{M_6^4}, \tag{2}$$

marking, respectively, the transition scale from 4d to 5d, from 5d to 6d, and finally from 6d to 7d.

In the absence of the 5d and 4d kinetic terms, the propagator on the codimension-1 brane is of the DGP form [5]

$$G_6^{(0)}(z-z') = \frac{1}{M_6^4}\int\frac{d\omega}{2\pi}\frac{e^{i\omega(z-z')}}{\omega^2 + q^2 + m_7\sqrt{q^2+\omega^2}}, \tag{3}$$

where q^α is the 5d momentum, and ω is the momentum associated with the z coordinate. The exact 6d propagator is then obtained by treating the 5d kinetic term as a perturbation localized at $z = 0$:

$$\begin{aligned} G_6(z,z') &= G_6^{(0)}(z-z') - M_5^3G_6^{(0)}(z)q^2G_6^{(0)}(-z') \\ &\quad + M_5^6G_6^{(0)}(z)q^4G_6^{(0)}(0)G_6^{(0)}(-z') + \ldots \\ &= G_6^{(0)}(z-z') - \frac{G_6^{(0)}(z)M_5^3q^2G_6^{(0)}(-z')}{1 + M_5^3q^2G_6^{(0)}(0)}. \end{aligned} \tag{4}$$

In particular the induced propagator on the codimension-2 brane is determined in terms of the integral of the higher-dimensional Green's function:

$$G_5^{(0)}(q^2) = G_6(0,0) = \frac{1}{M_5^3}\frac{1}{q^2 + g(q^2)}; \tag{5}$$

$$g(q^2) \equiv \frac{1}{M_5^3G_6^{(0)}(0)} = \frac{\pi m_6}{2}\frac{\sqrt{m_7^2 - q^2}}{\tanh^{-1}\left(\sqrt{\frac{m_7-|q|}{m_7+|q|}}\right)}. \tag{6}$$

(For $|q| > m_7$, we assume analytic continuation from the hyperbolic tangent to its trigonometric counterpart.)

Remarkably, the codimension-1 kinetic term makes the 5d propagator finite, thereby regulating the logarithmic divergence characteristic of pure codimension-2 branes. Indeed, $G_5^{(0)} \to M_7^{-5}\log(m_7q)$ as $M_6 \to 0$, and thus M_6 plays the role of a physical cutoff. As another check, note that in the limit $m_7 \to 0$ in which the bulk decouples, we recover the usual DGP result: $G_5^{(0)} \sim 1/(q^2 + m_6q)$.

It is straightforward to repeat the same steps to derive the induced 4d propagator on the codimension-3 brane.

II. Cascading Gravity.—We now proceed to the gravitational case. The 7d Einstein equations are given by

$$\begin{aligned} M_7^5G_{AB}^{(7)} &= -\delta(w)\{\delta_A^a\delta_B^bM_6^4G_{ab}^{(6)} + \delta(z)\delta_A^\alpha\delta_B^\beta M_5^3G_{\alpha\beta}^{(5)} \\ &\quad + \delta(z)\delta(y)\delta_A^\mu\delta_B^\nu[M_4^2G_{\mu\nu}^{(4)} + \Lambda g_{\mu\nu}]\}. \end{aligned} \tag{7}$$

The effective source therefore consists of induced gravity terms on each of the branes, as well as tension Λ on the codimension-3 brane.

In the weak-field approximation, the 7d line element can be written as $ds^2 = (\eta_{AB} + h_{AB})dx^Adx^B$. As shown in Appendix A, there is enough symmetry and gauge freedom to simplify the metric to the form

$$\begin{aligned} ds^2 &= [1 + \Phi(y,z,w)](dw^2 + dz^2 + dy^2) \\ &\quad - \frac{\Theta(w)}{2m_7}\partial_\alpha\Phi_0(y,z)dx^\alpha dw \\ &\quad + \left(1 - \frac{\Phi(y,z,w)}{4}\right)\eta_{\mu\nu}dx^\mu dx^\nu, \end{aligned} \tag{8}$$

where $\Phi_0(y,z) \equiv \Phi(y,z,w=0)$ is the induced profile on the codimension-1 brane. Here $\Theta(w)$ is the theta function: $\Theta(w) = +1$ for $w > 0$, and -1 for $w < 0$.

Substituting this ansatz into Einstein's equations (7), we find that Φ satisfies

$$\left(\Box_7 + \frac{\delta(w)}{m_7}\Box_6 - \frac{3}{5}\frac{\delta^2(z,w)}{m_7 m_6}\Box_5\right)\Phi = \frac{8}{5}\frac{\delta^3(y,z,w)}{M_7^5}\Lambda. \tag{9}$$

This equation is of the cascading form [12], as reviewed above. The asymptotically flat bulk solution is given by

$$\Phi(y,z,w) = e^{-|w|\sqrt{-\Box_6}}\Phi_0(y,z), \tag{10}$$

where the induced profile $\Phi_0(y,z)$ satisfies

$$\left(\Box_6 - m_7\sqrt{-\Box_6} - \frac{3}{5}\frac{\delta(z)}{m_6}\Box_5\right)\Phi_0 = \frac{8}{5}\frac{\delta^2(y,z)}{M_6^4}\Lambda. \tag{11}$$

To solve (11), we Fourier transform to momentum space and use the $6d$ and $5d$ Green's functions given, respectively, by (3) and (5). The result is

$$\Phi_0(y,z) = \int\frac{dq_y d\omega}{(2\pi)^2}\frac{e^{i\omega z}e^{iq_y y}g(q_y)\phi(q_y)}{\omega^2 + q_y^2 + m_7\sqrt{\omega^2+q_y^2}}, \tag{12}$$

where the Fourier transform of the codimension-2 profile, $\phi(q_y) = \int dy e^{-iq_y y}\Phi_0(z=0,y)$, satisfies

$$\left(\frac{3}{5}q_y^2 - g(q_y^2)\right)\phi(q_y) = \frac{8}{5M_5^3}\Lambda. \tag{13}$$

The solution to (13) can be expressed as the sum of a principal part $\mathcal{P}$ and two homogeneous modes:

$$\phi(q_y) = \frac{8\Lambda}{5M_5^3}\mathcal{P}\left[\frac{1}{\frac{3}{5}q_y^2 - g(q_y^2)}\right] + \sum_{\sigma=\pm}C_\sigma\delta(q_y - \sigma q_0),$$

where $\frac{3}{5}q_0^2 = g(q_0^2)$. Requiring the field Φ_0 to be real imposes $C_+ = C_- \equiv C$, while requiring Φ_0 to fall as $y \to 0$ sets $C = 0$. Using the resulting expression for $\phi(q_y)$ into (12) and then into (10), we obtain the final expression for the scalar potential $\Phi(y,z,w) = \frac{8\Lambda}{5M_6^4}\hat{\Phi}(y,z,w)$:

$$\hat{\Phi} = \int\frac{d\omega dq_y}{(2\pi)^2}\frac{e^{-|w|\sqrt{\omega^2+q_y^2}}e^{i\omega z}e^{iq_y y}}{\omega^2+q_y^2+m_7\sqrt{\omega^2+q_y^2}}\mathcal{P}\left[\frac{g(q_y)}{\frac{3}{5}q_y^2 - g(q_y)}\right]. \tag{14}$$

This is our main result. Thanks to the cascading mechanism, which has regularized all potential divergences, this solution is finite everywhere. Figure 2 shows that $\hat{\Phi}(y,z,w)$ is smooth everywhere and decreases with w.

As it stands, however, our framework has a ghost [8,9], as indicated by the poles at $q_y = \pm q_0$. There are two ways to resolve this issue. One can introduce sufficiently large tension on both the codimension-2 and -3 branes [11]: to remove the ghost, the codimension-2 tension should be $\gtrsim M_5^3 m_7^2$, whereas the corresponding bound on the codimension-3 tension is yet to be determined.

Alternatively, one can regularize codimension-2 and -3 branes and include the $6d$ Einstein-Hilbert term localized on these objects [9,12]. Following this route, we demonstrate in Appendix B that the poles do disappear, and that the profile for $\Phi(y,z,w)$ is qualitatively unchanged.

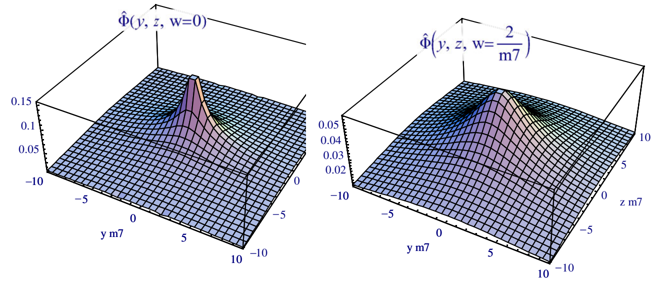

FIG. 2 (color online). Plot of the solution for the metric potential $\hat{\Phi}(y,z,w)$ for $w = 0$ and $w = 2m_7^{-1}$ in the case where $m_6 = m_7$.

III. Discussion.—In this Letter, we have shown that a 3-brane with tension remains flat in the $(6+1)$-dimensional cascading gravity framework. In the weak-field approximation, we have obtained a bulk solution which is nowhere singular and remains perturbative everywhere.

These properties crucially depend on the existence of parent branes with finite Planck masses. Indeed, our solution goes outside the perturbative regime and acquires divergences in the limit $M_5, M_6 \to 0$, consistent with [4].

We are currently extending our solutions to the nonlinear regime through numerical analysis. For now, we view the present results as a tantalizing first step towards realizing the idea of Rubakov and Shaposhnikov.

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Appendix A.—We show that the weak-field metric can be brought to the form (10) by symmetry and gauge freedom. In the de Donder gauge, $\partial_A h^A_B = \frac{1}{2}\partial_B h^C_C$, (7) reduces to

$$-\frac{M_7^5}{2}\Box_7\left(h_{AB} - \frac{1}{2}\eta_{AB}h^C_C\right) = \delta(w)(T^{(6)}_{ab} - M_6^4 G^{(6)}_{ab}),$$

where the effective stress energy on the codimension-1 brane, $T^{(6)}_{ab}$, includes contributions from the $5d$ and $6d$ induced gravity terms. Since there is no stress energy along the (a,w) and (w,w) directions, the corresponding equations are consistently satisfied by setting $h_{aw} = 0$ and $h_{ww} = h^c{}_c$, where $h^c{}_c$ is the $6d$ trace. It follows that the induced gauge choice in $6d$ is given by $\partial_a h^a_b = \partial_b h^c{}_c$; hence, the (a,b) components of (A1) reduce to

$$-\frac{M_7^5}{2}\Box_7(h_{ab} - \eta_{ab}h^c{}_c) = \delta(w)\frac{M_6^4}{2}(\Box_6 h_{ab} - \partial_a\partial_b h^c{}_c) + \delta(w)T^{(6)}_{ab}. \tag{A1}$$

To proceed further, it is convenient to decompose h_{ab} into its trace and transverse-traceless (TT) parts:

$$h_{ab} = h_{ab}^{6d\mathrm{TT}} + \frac{\partial_a \partial_b}{\Box_6} h^c{}_c. \tag{A2}$$

From (A1), the 6dTT components satisfy

$$-\frac{M_7^5}{2}\left(\Box_7 + \frac{\delta(w)}{m_7}\Box_6\right)h_{ab}^{6d\mathrm{TT}} = \delta(w)\left(T_{ab}^{(6)} - \frac{1}{5}\eta_{ab}T^{(6)} + \frac{1}{5}\frac{\partial_a\partial_b}{\Box_6}T^{(6)}\right). \tag{A3}$$

The symmetries of the problem allow a simple expression for the 5d components of the 6dTT part:

$$h_{\alpha\beta}^{6d\mathrm{TT}} = -\frac{1}{4}\Phi\eta_{\alpha\beta} - \left(\frac{\Box_5}{\Box_6} - \frac{5}{4}\right)\frac{\partial_\alpha\partial_\beta}{\Box_5}\Phi. \tag{A4}$$

This follows from setting $h_{\alpha\beta}^{5d\mathrm{TT}} = 0$, which is consistent with the equations of motion for the case of interest. Substituting into (A3), and using $T_{\alpha\beta}^{(5)} = -\delta^\mu{}_\alpha \delta^\nu{}_\beta \Lambda \eta_{\mu\nu}\delta(y)$, the resulting equation of motion for Φ agrees with (9).

We can now be explicit about the form of the various metric components. Combining (A2) and (A4), we get

$$h_{\alpha\beta} = -\frac{1}{4}\Phi\eta_{\alpha\beta} - \left(\frac{\Box_5}{\Box_6} - \frac{5}{4}\right)\frac{\partial_\alpha\partial_\beta}{\Box_5}\Phi + \frac{\partial_\alpha\partial_\beta}{\Box_6}h^c{}_c. \tag{A5}$$

And since everything is independent of x^μ, we get $h_{y\mu} = 0$ and $h_{\mu\nu} = -\frac{1}{4}\Phi\eta_{\mu\nu}$. Similarly, from (A2) we obtain

$$h_{yz} = \frac{\partial_y\partial_z}{\Box_6}(h^c{}_c - \Phi); \qquad h_{zz} = \frac{\partial_z^2}{\Box_6}(h^c{}_c - \Phi) + \Phi;$$
$$h_{yy} = \frac{\partial_y^2}{\Box_6}(h^c{}_c - \Phi) + \Phi. \tag{A6}$$

This is equivalent to (8) after a small diffeomorphism.

Appendix B.—One way to cure the ghost of higher-codimension DGP models [8,9] is to consider a higher-dimensional Einstein-Hilbert term localized on the regularized brane [9,12]. Following this prescription, we will show that the solution remains finite everywhere.

When adding a 6d Einstein-Hilbert term on the regularized 4-brane, on the top of the usual 5d Einstein-Hilbert term of the form "$\Box_5 h_{\alpha\beta}$" we must consider excitations of transverse modes along the extra dimensions as well as the higher-dimensional mode h_{zz}. In the thin-brane limit, however, the excitations along the extra dimension become very massive, so that any term containing z derivatives can be neglected. Meanwhile, h_{zz} survives in the limit; see [12] for details.

In the 7d de Donder gauge, the Einstein equations are the same as in (A1). Setting $h_{aw} = 0$ and $h_{ww} = h^c{}_c$, we have

$$-\frac{M_7^5}{2}\left(\Box_7 + \frac{\delta(w)}{m_7}\Box_6\right)h_{ab} = \delta(w)\left(T_{ab}^{(6)} - \frac{1}{5}T^{(6)}\eta_{ab}\right) \tag{B1}$$

with $T_{z\alpha}^{(6)} = 0$, $T_{zz}^{(6)} = M_5^3\delta(z)R_5/2$, and

$$T_{\alpha\beta}^{(6)} = -M_5^3\delta(z)[G_{\alpha\beta}^{(5)} + \tfrac{1}{2}(\Box_5 h_{zz}\eta_{\alpha\beta} - \partial_\alpha\partial_\beta h_{zz})] - \delta(z)\delta(y)\Lambda\eta_{\mu\nu}\delta_\alpha^\mu\delta_\beta^\nu. \tag{B2}$$

Using this in the 6d part of the Einstein equations, we get $h_{zz} = -\psi$, $\Box_5 h_{yy} = -4\Box_5\psi + \partial_y^2 h^\alpha{}_\alpha$, $h_{\mu y} = 0$, and $\Box_5 h_{\mu\nu} = \Box_5\psi\eta_{\mu\nu} + \partial_\mu\partial_\nu h^\alpha{}_\alpha$, with

$$\left[\Box_7 + \frac{\delta(w)}{m_7}\Box_6 + \frac{\delta^{(2)}(w,z)}{m_7 m_6}\Box_5\right]\psi = \frac{2}{5}\frac{\delta^{(3)}(w,z,y)}{M_7^5}\Lambda. \tag{B3}$$

We notice that the kinetic term for ψ is now everywhere positive, signaling that the ghost has been cured. Equation (B3) is similar to (9) for Φ, except for a redefinition of m_6 and M_7. The profile for $\psi(y,z,w) = -\frac{2\Lambda}{5M_6^4}\hat{\Psi}(y,z,w)$,

$$\hat{\Psi} = \int\frac{d\omega dq_y}{(2\pi)^2}\frac{e^{-|w|\sqrt{\omega^2+q_y^2}}e^{i\omega z}e^{iq_y y}}{\omega^2 + q_y^2 + m_7\sqrt{\omega^2+q_y^2}}\frac{g(q_y)}{q_y^2 + g(q_y)}, \tag{B4}$$

is similar to that of $\hat{\Phi}$, and, in particular, is free of divergences. The static solution for a codimension-3 brane with tension remains therefore well defined, at least in the weak-field approximation, in a ghost-free setup.

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