

Entropic Entanglement Criteria for Continuous Variables

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We derive several entanglement criteria for bipartite continuous variable quantum systems based on the Shannon entropy. These criteria are more sensitive than those involving only second-order moments, and are equivalent to well-known variance product tests in the case of Gaussian states. Furthermore, they involve only a pair of quadrature measurements, and will thus prove extremely useful in the experimental identification of entanglement.

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Quantum entanglement is the property that differentiates quantum-mechanical systems from classical ones. As such, the detection and characterization of quantum entanglement is one of the prominent goals in quantum information. In the discrete variable case, many detection schemes for entanglement have been proposed (see [1] for a review). In the continuous variable (CV) case, detection of entanglement is more challenging due to the complicated Hilbert space structure, and many tests only detect entanglement that appears in the second-order moments [2–7], which is completely adequate for the case of Gaussian states. However, non-Gaussian states and processes have been shown to not just enhance certain quantum information protocols such as teleportation [8,9], but in fact be necessary for certain tasks, such as universal quantum computing [10,11] and entanglement distillation [12,13]. Towards the detection of CV entanglement in general, Shchukin and Vogel have derived an infinite hierarchy of conditions for positive partial transpose involving higher-order moments [14]. Although powerful, these conditions may not always be experimentally convenient [15].

Here we derive several entropic entanglement criteria for CVs. In contrast to previous work based on quantum-mechanical generalizations of entropy functions [16–19], our criteria involve the Shannon entropy of probability distributions of a pair of complementary quadrature measurements. We will show that these conditions detect entanglement in many states that any second-order test will not. A first set of inequalities is most sensitive, but is valid only for pure states. Inspired by previous work in discrete variables [20], we use the entropic uncertainty relation attained in [21] for probability distributions of complementary quantum observables, to derive a second set of inequalities. These have the distinct advantage that they can be extended to include mixed bipartite CV states. These inequalities are more sensitive than the usual criteria based on second-order moments [2–5], and are equivalent to a well-known variance product criteria [4] in the case of bipartite Gaussian states. At the same time they are no more experimentally demanding than the widely adopted tests [3–5].

As in other CV inseparability criteria [3–5], we consider the global operators

$$r_{\pm} = r_1 \pm r_2 \quad \text{and} \quad \mathbf{s}_{\pm} = \mathbf{s}_1 \pm \mathbf{s}_2, \quad (1)$$

where $r_j = \cos\theta_j \mathbf{x}_j + \sin\theta_j \mathbf{p}_j$, $\mathbf{s}_j = \cos\theta_j \mathbf{p}_j - \sin\theta_j \mathbf{x}_j$, and \mathbf{x}_j and \mathbf{p}_j are the usual canonical variables satisfying $[\mathbf{x}_j, \mathbf{p}_i] = i\delta_{ij}$, and $j, i = 1, 2$ refers to each subsystem of a bipartite state. Note also that the operators r_j and \mathbf{s}_j satisfy $[r_j, \mathbf{s}_i] = i\delta_{ij}$. The entropy associated to a measurement of r is given by the Shannon entropy

$$H[R] = - \int dr R(r) \ln R(r), \quad (2)$$

where $R(r)$ is the probability distribution associated to the measurement of r , and similarly for $H[S]$.

We will derive inseparability criteria of the form

$$H[R_{\pm}] + H[S_{\mp}] \geq c, \quad (3)$$

where R_{\pm} and S_{\mp} are the probability distributions associated to measurement of r_{\pm} and \mathbf{s}_{\mp} , respectively, and $c > 0$ is a real constant. Any separable state obeys inequality (3), while entangled states may not. For example, the left side of Eq. (3) vanishes for the common eigenstates of r_{-} and \mathbf{s}_{+} or r_{+} and \mathbf{s}_{-} , which correspond EPR-like states (note that $[r_{+}, \mathbf{s}_{-}] = [r_{-}, \mathbf{s}_{+}] = 0$).

Let us first derive an inequality for the case of pure states. We will then extend our results to include mixed states. A separable pure state can be written in the form $|\psi_1\rangle \otimes |\psi_2\rangle$, and has a corresponding wave function $\Psi(r_1, r_2) = \psi_1(r_1)\psi_2(r_2)$. Using Eq. (1) and changing variables gives

$$\Psi(r_{+}, r_{-}) = \frac{1}{\sqrt{2}} \psi_1\left(\frac{r_{+} + r_{-}}{2}\right) \psi_2\left(\frac{r_{+} - r_{-}}{2}\right). \quad (4)$$

The probability distribution associated to the measurement of r_{\pm} is given by

$$R_{\pm} = \frac{1}{2} \int dr_{\mp} R_1\left(\frac{r_+ + r_-}{2}\right) R_2\left(\frac{r_+ - r_-}{2}\right) \\ = \int dr R_1(r) R_2(\mp r \pm r_{\pm}) = R_1 * R_2^{(\pm)}, \quad (5)$$

where $R_j(r) = |\psi_j(r)|^2$, the symbol “*” denotes convolution and $R_2^{(+)} \equiv R_2(r)$, $R_2^{(-)} \equiv R_2(-r)$. Using the entropy power inequality [22,23]

$$\exp(2H[A * B]) \geq \exp(2H[A]) + \exp(2H[B]), \quad (6)$$

and also the fact that the Shannon entropy is invariant under reflections [22], we have

$$H[R_{\pm}] \geq \frac{1}{2} \ln\{\exp(2H[R_1]) + \exp(2H[R_2])\}. \quad (7)$$

We arrive at an equivalent inequality for $H[S_{\mp}]$:

$$H[S_{\mp}] \geq \frac{1}{2} \ln\{\exp(2H[S_1]) + \exp(2H[S_2])\}. \quad (8)$$

Combining (7) and (8), we have

$$H[R_{\pm}] + H[S_{\mp}] \geq \frac{1}{2} \ln\left\{ \sum_{i,j=1,2} e^{(2H[R_i] + 2H[S_j])} \right\}. \quad (9)$$

Equation (9) gives two inequalities that are satisfied by separable pure states. Violation of either inequality (9) is then a sufficient condition for entanglement.

We will now show that it is possible to arrive at a weaker pair of inequalities, and then extend them to include mixed states. We make use of the entropic uncertainty relation for probability distributions R and S associated to the measurement of complementary quantum observables \mathbf{r} and \mathbf{s} such that $[\mathbf{r}, \mathbf{s}] = i$ [21]:

$$H[R] + H[S] \geq \ln \pi e, \quad (10)$$

specialized for $R = R_j$ and $S = S_j$ with $j = 1, 2$. This gives $\exp(2H[R_j] + 2H[S_j]) \geq (\pi e)^2$, which leads to

$$H[R_{\pm}] + H[S_{\mp}] \geq \frac{1}{2} \ln\left\{ 2(\pi e)^2 + \sum_{i \neq j} e^{(2H[R_i] + 2H[S_j])} \right\}. \quad (11)$$

Using relation (10) again, one obtains

$$H[R_{\pm}] + H[S_{\mp}] \geq \frac{1}{2} \ln\{2(\pi e)^2(1 + \cosh \Delta H)\}, \quad (12)$$

where $\Delta H = 2H[S_2] - 2H[S_1]$. Since the hyperbolic cosine is lower-bounded by 1, we have

$$H[R_{\pm}] + H[S_{\mp}] \geq \ln 2\pi e. \quad (13)$$

The term on the right side is a state-independent lower bound for $H[R_{\pm}] + H[S_{\mp}]$ of any separable pure state.

To extend the inequalities (13) to include mixed states, we use the fact that any bipartite separable state ρ can be decomposed into a convex sum of pure states

$$\rho = \sum_k \lambda_k |\psi_{1k}\rangle\langle\psi_{1k}| \otimes |\psi_{2k}\rangle\langle\psi_{2k}|, \quad (14)$$

where $\lambda_k \geq 0$ and $\sum_k \lambda_k = 1$. The probability distributions associated to a measurement r_{\pm} is $R_{\pm} = \sum_k \lambda_k R_{k\pm}$, where $R_{k\pm}$ is the probability to detect r_{\pm} for each pure state in the decomposition (14). The concavity of the Shannon entropy [23] gives

$$H[R_{\pm}] \geq \sum_k \lambda_k H[R_{k\pm}], \quad (15)$$

and likewise for $H[S_{\mp}]$. As $H[R_{k\pm}] = H[R_{k1} * R_{k2}]$, inequality (6) gives

$$H[R_{\pm}] \geq \sum_k \frac{\lambda_k}{2} \ln(e^{2H[R_{k1}]} + e^{2H[R_{k2}]}) \quad (16)$$

A similar condition holds for $H[S_{\mp}]$. Summing these two inequalities, and using the fact that the left side of (9) is lower-bounded by $\ln 2\pi e$, gives

$$H[R_{\pm}] + H[S_{\mp}] \geq \sum_k \frac{\lambda_k}{2} \ln\left\{ \sum_{i,j=1,2} e^{(2H[R_{ki}] + 2H[S_{kj}])} \right\} \\ \geq \sum_k \lambda_k \ln(2\pi e) = \ln 2\pi e, \quad (17)$$

which is identical to (13). Thus, inequalities (13) are satisfied by both pure and mixed separable states. We note also that one can take the supremum of the first inequality (17) over all possible decompositions of the mixed state ρ to arrive at a stronger inequality. However, this is not suitable for experimental purposes.

We also note that inequalities (13) can also be obtained using the positive partial transpose criterion [2] as we will now show. First, we note that the marginal distributions under partial transposition are

$$\tilde{R}_{\pm} = R_{\pm} \quad \text{and} \quad \tilde{S}_{\pm} = S_{\mp}. \quad (18)$$

Noting that $[\mathbf{r}_{\pm}, \mathbf{s}_{\pm}] = 2i$, the relations in (18) imply that any separable state must verify the uncertainty relation $H[\tilde{R}_{\pm}] + H[\tilde{S}_{\pm}] \geq \ln 2\pi e$, which leads directly to inequalities (13). Furthermore, we see in (18) that the partial transposition interchanges the variables s_+ and s_- , which is the key to obtain entanglement criteria based on uncertainty inequalities that can arise from the noncommutativity of \mathbf{r}_{\pm} and \mathbf{s}_{\pm} .

An upper bound to the left side of (13) can be obtained by considering that the Shannon entropy of a continuous variable with variance σ^2 is maximized when the probability distribution is Gaussian, for which $H_{\text{Gauss}} = \ln\sqrt{2\pi e\sigma^2}$ [22,23]. Then,

$$\ln 2\pi e \sigma_{\pm} \delta_{\mp} \geq H[R_{\pm}] + H[S_{\mp}] \geq \ln 2\pi e, \quad (19)$$

where σ_{\pm}^2 and δ_{\mp}^2 are the variances of R_{\pm} and S_{\mp} , respectively. This upper limit is reached for Gaussian states, in which case we recover the Mancini-Giovannetti-Vitali-Tombesi (MGVT) product inequality [4]

$$\sigma_{\pm} \delta_{\mp} \geq 1. \quad (20)$$

The left side of the double inequality (19) proves that the conditions (13) are more sensitive than the variance product criteria, with equivalence in the case of Gaussian states. This is in accord with the extremality of entangled Gaussian states [24].

In the definition of the operators (1) we included the parameter θ_j to account for local rotations of the quadrature measurements. To successfully employ the entropic criteria, it is necessary to find suitable rotated quadrature operators, parametrized by angles θ_1 and θ_2 . In addition, one can optimize over local squeezing parameters, which can be included *a posteriori* [7]. So, in order to complete the analysis of the effect of real linear canonical transformations over the single mode quadrature operators [that form the real symplectic group $Sp(2, \mathbb{R})$] let us consider now the effect of local squeezing, which can be accounted for by redefining the rotated operators as

$$r'_{\pm} = a_1 r_1 \pm a_2 r_2 \quad \text{and} \quad s'_{\pm} = \frac{1}{a_1} s_1 \pm \frac{1}{a_2} s_2. \quad (21)$$

Substituting these operators, inequality (9) becomes

$$H[R'_{\pm}] + H[S'_{\pm}] \geq \frac{1}{2} \ln \left\{ \sum_{i,j=1,2} e^{(2H[R_i] + 2H[S_j] + 2\ln(a_i/a_j))} \right\}, \quad (22)$$

and applying the entropic uncertainty relation (10) results in

$$H[R'_{\pm}] + H[S'_{\pm}] \geq \ln(2\pi e), \quad (23)$$

where R'_{\pm} and S'_{\pm} are the probability distributions for measurements of the operators (21). It is straightforward to show that both entropy inequalities (22) and (23) reduce to (9) and (13) when $a_1 = a_2$, which demonstrates that these inequalities are invariant to equal amounts of local squeezing.

Examples.—Inequalities (9) and (13) are more sensitive than criteria involving sums or products of variances. We will now illustrate this point with several examples of non-Gaussian states. Let us first consider a non-Gaussian wave function of the form

$$\eta(r_1, r_2) = \frac{(r_1 + r_2)}{\sqrt{\pi\sigma_- \sigma_+^3}} e^{-(r_1+r_2)^2/4\sigma_+^2} e^{-(r_1-r_2)^2/4\sigma_-^2}. \quad (24)$$

This state is nonseparable for all values of σ_{\pm} . For this state both the Simon PPT criteria [2], which is a necessary and sufficient condition to detect entanglement in Gaussian states, and the MGVT criteria with $\theta_1 = \theta_2 = 0$, detect entanglement provided $\sigma_-/\sigma_+ > \sqrt{3} \approx 1.732$ or $\sigma_-/\sigma_+ < 1/\sqrt{3} \approx 0.577$. With $\theta_1 = \theta_2 = 0$, the inseparability criteria (13) gives $H[R_{\pm}] + H[S_{\pm}] = \ln(4\pi e^{\gamma} \sigma_{\mp}/\sigma_{\pm})$, where $\gamma = 0.577 \dots$ is Euler's constant. Entanglement is detected provided $\sigma_-/\sigma_+ < e^{(1-\gamma)}/2 \approx 0.763$ or $\sigma_-/\sigma_+ > 2/e^{(1-\gamma)} \approx 1.310$. Thus, the entropy criterion (13) is more sensitive than the Simon and MGVT

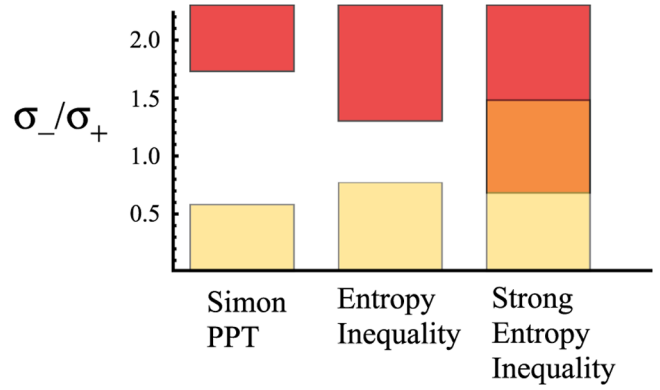


FIG. 1 (color online). Pictorial representation of three inseparability criteria for the pure state (24) as a function of σ_-/σ_+ . The criteria called “entropy inequality” and “strong entropy inequality” correspond, respectively, to Eqs. (13) and (9). The shaded regions correspond to the intervals of σ_-/σ_+ where each criteria detect entanglement and blank regions where they do not. Both criteria presented here are stronger than the Simon PPT condition [2].

conditions. Numerical results show that the ranges in which the pair of inequalities (9) detect entanglement overlap, indicating that they always detect entanglement in the state (24). A pictorial representation of these results is shown in Fig. 1.

We tested these criteria numerically for a number of combinations of low-order Fock states. A particularly interesting example is the state $|\phi\rangle = |0, 0\rangle/\sqrt{2} + |2, 0\rangle/2 + |0, 2\rangle/2$, which is undetectable with any second-order criterion [25]. Entanglement is detectable using the stronger entropic inequality (9) with $\theta_1 = -\pi/4$ and $\theta_2 = \pi/4$. We also tested “ $N00N$ ” states of the form $(|N, 0\rangle + |0, N\rangle)/\sqrt{2}$, where $|N\rangle$ is an N -photon Fock state. None of these states is detected by the Simon PPT criteria [2] nor inequalities (13). The strong criteria (9) detects entanglement up to $N = 5$ with $\theta_1 = \theta_2 = 0$ for all values of N except $N = 2$, for which $\theta_1 = 0$ and $\theta_2 = \pi/2$ were used.

To further test these criteria we generated uniform random pure states [26] of the form $|\psi\rangle = \sum_{n,m=0}^D C_{nm} |n, m\rangle$, and tested inequalities (9) and (13) and also the MGVT

TABLE I. Results for random states $|\psi\rangle$ (see text). n_{strong} , n_{weak} , and n_{MGVT} are the percentage of states detected by inequalities (9), (13), and (20), respectively. In all cases, the angles θ_1 and θ_2 were scanned in intervals of $\pi/4$.

Number of states	D	n_{strong}	n_{weak}	n_{MGVT}
6000	2	74.4%	17.3%	9.9%
1600	3	86.3%	0.5%	0.2%
800	4	84.9%	0%	0%
720	5	81.0%	0%	0%
120	7	62.5%	0%	0%

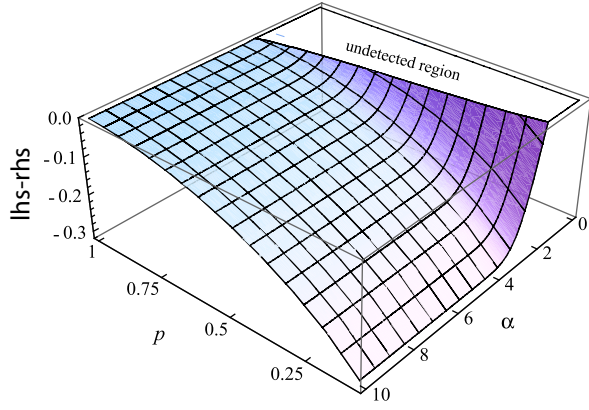


FIG. 2 (color online). Violation of entanglement criteria (13) for the dephased cat state (25). The vertical axis is the difference of the left-hand side (lhs) and right-hand side (rhs) of (13). See text for details.

criteria, while varying angles θ_1 and θ_2 in intervals of $\pi/4$. Results are summarized in Table I. The strong inequality (9) detected more than 62% of the randomly generated states up to $D = 7$ (the vanishing fraction of randomly generated separable states would only increase this value).

Criteria (13) also applies to mixed states. Let us consider a dephased cat state characterized by the parameter $0 \leq p \leq 1$, given by

$$\rho = N(\alpha)\{|\alpha, \alpha\rangle\langle\alpha, \alpha| + |-\alpha, -\alpha\rangle\langle-\alpha, -\alpha| - (1-p)(|\alpha, \alpha\rangle\langle-\alpha, -\alpha| + |-\alpha, -\alpha\rangle\langle\alpha, \alpha|)\}, \quad (25)$$

where $N(\alpha)$ is a normalization constant. This state is separable only when $p = 1$, and is undetected by any second-order criteria for any value of p . The entanglement criteria (13) is shown in Fig. 2 as a function of α and p for real α . Using $\theta_1 = \theta_2 = 0$, the sum $H[R_-] + H[S_+]$ is less than $\ln 2\pi e$, and thus entanglement is detected, for a large range of α and p .

Let us briefly discuss the application of these entropic criteria in an experimental setting. For fixed values θ_1 and θ_2 of the local rotations the Shannon entropies $H[R_\pm]$ and $H[S_\mp]$ can be calculated using the marginal probability distributions R_\pm and S_\mp . These can be determined directly via measurement of r_\pm and s_\mp , or calculated from the joint probability distributions $R(r_1, r_2)$ and $S(s_1, s_2)$. We stress that the sole determination of these probability distributions does not allow one to calculate arbitrary moments involving products of the r_j and s_j operators. Hence, these measurements alone are not enough to determine all the second-order moments required to evaluate even the second-order criterion of Simon [2]. The higher-order criteria in [14] requires even more involved measurement schemes [15]. In summary, the evaluation of our entropic criteria requires the same experimental resources as those

required to evaluate the commonly employed second-order inequalities in [3–5], while providing a more sensitive entanglement test. We thus expect that the inseparability tests presented here will be of great use in experimental settings.

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