Holographic Superconductivity in M Theory

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Using seven-dimensional Sasaki-Einstein spaces we construct solutions of $D = 11$ supergravity that are holographically dual to superconductors in three spacetime dimensions. Our numerical results indicate a new zero temperature solution dual to a quantum critical point.

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Introduction.—The anti–de Sitter/conformal field theory (AdS/CFT) correspondence provides a powerful framework for studying strongly coupled quantum field theories using gravitational techniques. It is an exciting possibility that these techniques can be used to study classes of superconductors which are not well described by more standard approaches [1–3].

The basic setup requires that the CFT has a global Abelian symmetry corresponding to a massless gauge field propagating in the AdS space. We also require an operator in the CFT that corresponds to a scalar field that is charged with respect to this gauge field. Adding a black hole to the AdS space describes the CFT at finite temperature. One then looks for cases where there are high temperature black hole solutions with no charged scalar hair but below some critical temperature black hole solutions with charged scalar hair appear and moreover dominate the free energy. Since we are interested in describing superconductors in flat spacetime we consider black holes with planar symmetry. In order to obtain a critical temperature, conformal invariance then implies that another scale needs to be introduced. This is achieved by considering electrically charged black holes which corresponds to studying the dual CFT at finite chemical potential.

Precisely this setup has been studied using a phenomenological theory of gravity in $D = 4$ coupled to a single charged scalar field and it has been shown that, for certain parameters, the system manifests superconductivity in three spacetime dimensions, in the above sense [3]. It is important to go beyond such models and construct solutions in the context of string or M theory so that there is a consistent underlying quantum theory and CFT dual. Also, as we shall see, the behavior of the string or M-theory solutions will differ substantially from that of the phenomenological model [3] at low temperature. It was shown in [4] that the $D = 4$ phenomenological models of [3] arise, at the linearized level, after Kaluza-Klein (KK) reduction of $D = 11$ supergravity on a seven-dimensional Sasaki-Einstein space SE_7. Here we go beyond this linearized approximation by working with a consistent truncation of the $D = 4$ KK reduced theory presented in [5]. The truncation is consistent in the sense that any solution of this $D = 4$ theory, combined with a given SE_7 metric, gives rise to an exact solution of $D = 11$ supergravity. Here we shall use this $D = 4$ theory to construct exact solutions of $D = 11$ supergravity that correspond to holographic superconductivity.

The KK truncation.—We begin by recalling that any SE_7 metric can, locally, be written as a fibration over a six-dimensional Kähler-Einstein space, KE_6:

$$ds^2(SE_7) \equiv ds^2(KE_6) + \eta \otimes \eta. \tag{1}$$

Here η is the one-form dual to the Reeb Killing vector satisfying $d\eta = 2J$ where J is the Kähler form of KE_6. We denote the $(3, 0)$ form defined on KE_6 by Ω. For a regular or quasiregular SE_7 manifold, the orbits of the Reeb vector all close, corresponding to compact $U(1)$ isometry, and the KE_6 is a globally defined manifold or orbifold, respectively. For an irregular SE_7 manifold, the Reeb vector generates a noncompact $\mathbb{R}$ isometry and the KE_6 is only locally defined.

In the KK ansatz of [5] the $D = 11$ metric is written

$$\begin{aligned} ds^2 = {}& e^{-6U-V} ds_4^2 + e^{2U} ds^2(KE_6) \\ & + e^{2V}(\eta + A_1) \otimes (\eta + A_1) \end{aligned} \tag{2}$$

while the four-form is written

$$\begin{aligned} G_4 = {}& 6e^{-18U-3V}(\epsilon + h^2 + |\chi|^2)\mathrm{vol}_4 + H_3 \wedge (\eta + A_1) \\ & + H_2 \wedge J + dh \wedge J \wedge (\eta + A_1) + 2hJ \wedge J \\ & + \sqrt{3}[\chi(\eta + A_1) \wedge \Omega - \tfrac{i}{4} D\chi \wedge \Omega + \text{c.c.}], \end{aligned} \tag{3}$$

where ds_4^2 is a four-dimensional metric (in Einstein frame), U, V, h are real scalars, and χ is a complex scalar defined on the four-dimensional space. Furthermore, also defined on this four-dimensional space are A_1 a one-form potential, with field strength $F_2 \equiv dA_1$, two-form and three-form field strengths H_2 and H_3, related to one-form and two-form potentials via $H_3 = dB_2$ and $H_2 = dB_1 + 2B_2 + hF_2$. Finally $D\chi \equiv d\chi - 4iA_1\chi$.

This is a consistent KK truncation of $D = 11$ supergravity in the sense that if the equations of motion for the $4d$ fields ds_4^2, U, V, A_1, H_2, H_3, h, χ as given in [5] are satisfied then so are the $D = 11$ equations. The $D = 4$ equations of motion admit a vacuum solution with vanishing matter fields which uplifts to the $D = 11$ solution:

$$ds^2 = \tfrac{1}{4}ds^2(\mathrm{AdS}_4) + ds^2(\mathrm{SE}_7), \quad G_4 = \epsilon \tfrac{3}{8}\mathrm{vol}(\mathrm{AdS}_4), \quad (4)$$

where $ds^2(\mathrm{AdS})_4$ is the standard unit radius metric. When $\epsilon = +1$, this $\mathrm{AdS}_4 \times \mathrm{SE}_7$ solution is supersymmetric and describes $M2$ branes sitting at the apex of the Calabi-Yau fourfold (CY_4) cone whose base space is given by the SE_7. When $\epsilon = -1$ the solution is a "skew-whiffed" $\mathrm{AdS}_4 \times \mathrm{SE}_7$ solution, which describes anti-$M2$ branes sitting at the apex of the CY_4 cone. These solutions break all of the supersymmetry except for the special case when the SE_7 is the round seven-sphere, S^7, in which case it is maximally supersymmetric. Note that the skew-whiffed solutions with $\mathrm{SE}_7 \neq S^7$ are perturbatively stable [6], despite the absence of supersymmetry. Thus such backgrounds should be dual to three-dimensional CFTs at least in the strict $N = \infty$ limit. We are most interested in the skew-whiffed case because it is for that case that the operator dual to χ has scaling dimensions $\Delta = 1$ or 2 [5] and, based on the work of [4], is when we expect holographic superconductivity.

The $D = 4$ equations of motion can be derived from a four-dimensional action given in [5]. It is convenient to work with an action that is obtained after dualizing the one-form B_1 to another one-form $\tilde{B}_1$ and the two-form B_2 to a scalar a as explained in Sec. 2.3 of [5]. The dual fields are related to the original fields via

$$H_3 = -e^{-12U} * [da + 6(\tilde{B}_1 - \epsilon A_1) - \tfrac{3}{4}i(\chi^* D\chi - \chi D\chi^*)],$$
$$H_2 = (4h^2 + e^{4U+2V})^{-1}[2h - e^{2U+V}*](\tilde{H}_2 + h^2 F_2), \quad (5)$$

where $\tilde{H}_2 \equiv d\tilde{B}_1$. We now restrict to the (skew-whiffed) case $\epsilon = -1$. For this case we can make the following additional truncation of the $D = 4$ theory:

$$a = h = 0, \quad V = -2U, \quad A_1 = -\tilde{B}_1, \quad e^{6U} = 1 - \tfrac{3}{4}|\chi|^2. \quad (6)$$

One can show that provided that we restrict to configurations satisfying $F_2 \wedge F_2 = 0$ we obtain equations of motion that can be derived from the action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\bigg[R - \frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + \Big(1 - \frac{1}{2}|\hat{\chi}|^2\Big)^{-2} \times \Big(-|D\hat{\chi}|^2 + 24\Big(1 - \frac{2}{3}|\hat{\chi}|^2\Big)\Big)\bigg], \quad (7)$$

where $D\hat{\chi} \equiv d\hat{\chi} - 2i\hat{A}_1\hat{\chi}$, and we have defined $\hat{A}_1 \equiv 2A_1$, $\hat{\chi} \equiv (3/2)^{1/2}\chi$. Linearizing in the complex scalar $\hat{\chi}$, this gives the action considered in [3] (with their $L = 1/2$ and their $q = 2$). This nonlinear action is in the class considered in [7] and in addition to the AdS_4 vacuum with $\hat{A}_1 = 0$ and $\hat{\chi} = 0$, which uplifts to (4), it also admits AdS_4 vacuua with $\hat{A}_1 = 0$ and constant $|\hat{\chi}| = 1$, which uplift to the $D = 11$ solutions [8] found in [11].

Black hole solutions.—The key result of the last section is that any solution to the $D = 4$ equations of motion of the action (7) with $\hat{F} \wedge \hat{F} = 0$, gives an exact solution of $D = 11$ supergravity for any SE_7 metric. To find solutions relevant for studying superconductivity via holography we consider the following ansatz:

$$ds^2 = -ge^{-\beta}dt^2 + g^{-1}dr^2 + r^2(dx^2 + dy^2),$$
$$\hat{A}_1 = \hat{\phi}dt, \qquad \hat{\chi} \equiv \sigma \in \mathbb{R}, \quad (8)$$

where g, β, $\hat{\phi}$, and σ are all functions of r only. Being purely electrically charged this satisfies the $\hat{F} \wedge \hat{F} = 0$ condition. After substituting into the equations of motion arising from (7), we are led to ordinary differential equations which can also be obtained from the action obtained by substituting the ansatz directly into (7):

$$S = c\int dr r^2 e^{-\beta/2}\bigg[-g'' + g'\Big(\frac{3}{2}\beta' - \frac{4}{r}\Big) + g\Big(\beta'' - \frac{1}{2}(\beta')^2 + 2\frac{\beta'}{r} - \frac{2}{r^2}\Big) + \frac{1}{2}e^{\beta}(\hat{\phi}')^2 + \Big(1 - \frac{1}{2}\sigma^2\Big)^{-2}\Big(-g(\sigma')^2 + 4g^{-1}e^{\beta}\hat{\phi}^2\sigma^2 + 24\Big(1 - \frac{2}{3}\sigma^2\Big)\Big)\bigg], \quad (9)$$

where $c = (16\pi G)^{-1}\int dtdxdy$.

We next observe that the system admits the following exact AdS Reissner-Nordström type solution $\sigma = \beta = 0$:

$$g = 4r^2 - \frac{1}{r}\Big(4r_+^3 + \frac{\alpha^2}{r_+}\Big) + \frac{\alpha^2}{r^2}, \quad \hat{\phi} = \alpha\Big(\frac{1}{r_+} - \frac{1}{r}\Big) \quad (10)$$

for some constants α, r_+. The horizon is located at $r = r_+$ and for large r it asymptotically approaches $1/4$ of a unit radius AdS_4 [see (4)]. This solution should describe the high temperature phase of the superconductor.

We are interested in finding more general black hole solutions with charged scalar hair, $\sigma \neq 0$. Let us examine the equations at the horizon and at infinity. At the horizon $r = r_+$ we demand that $g(r_+) = \hat{\phi}(r_+) = 0$. One then finds that the solution is specified by 4 parameters at the horizon r_+, $\beta(r_+)$, $\hat{\phi}'(r_+)$, $\sigma(r_+)$. At $r = \infty$ we have the asymptotic expansion,

$$\beta = \beta_a + \ldots, \frac{\sigma}{\sqrt{8\pi G}} = \frac{\sigma_1}{r} + \frac{\sigma_2}{r^2} + \ldots,$$
$$\frac{\hat{\phi}}{\sqrt{16\pi G}} = e^{-\beta_a/2}\Big(\hat{\mu} - \frac{\hat{q}}{r}\Big) + \ldots \quad (11)$$
$$e^{-\beta}g = e^{-\beta_a}\bigg(4r^2 - \frac{8\pi G(m + \frac{4}{3}\sigma_1\sigma_2)}{r}\bigg) + \ldots$$

determined by the data β_a, $\sigma_{1,2}$, m, $\hat{\mu}$, $\hat{q}$. The scaling

$$r \to ar, \quad (t, x, y) \to a^{-1}(t, x, y), \quad g \to a^2 g, \quad \hat{\phi} \to a\hat{\phi} \quad (12)$$

leaves the metric, A_1, and all equations of motion invariant.

Action and thermodynamics.—We analytically continue by defining $\tau \equiv it$. The temperature of the black hole is $T = e^{\beta_a/2}/\Delta\tau$ where $\Delta\tau$ is fixed by demanding regularity of the Euclidean metric at $r = r_+$. We find:

$$T = \frac{r_+ e^{(\beta_a - \beta)/2}}{4\pi}\left[\frac{12(1-\frac{2}{3}\sigma^2)}{(1-\frac{1}{2}\sigma^2)^2} - \frac{1}{4}e^{\beta}\hat{\phi}'^2\right]_{r=r_+}. \quad (13)$$

Defining $I \equiv -iS$, we can calculate the on-shell Euclidean action $I_{\rm OS}$

$$I_{\rm OS} = \frac{\Delta\tau \mathrm{vol}_2}{16\pi G}\int_{r_+}^{\infty} dr[r^2 e^{-\beta/2}(g' - g\beta' - e^{\beta}\hat{\phi}\hat{\phi}')]'$$
$$= \frac{\Delta\tau \mathrm{vol}_2}{16\pi G}\int_{r_+}^{\infty} dr[2rge^{-\beta/2}]', \quad (14)$$

where $\mathrm{vol}_2 \equiv \int dxdy$. The latter expression only gets contributions from the on-shell functions at $r = \infty$ since $g(r_+) = 0$, while the former expression gets contributions from $r = r_+$ and $r = \infty$. The on-shell action diverges and we need to regulate by adding appropriate counterterms. We define $I_{\rm tot} \equiv I + I_{\rm ct}$ and, for simplicity, we will focus on the following counterterm action $I_{\rm ct}$:

$$I_{\rm ct} = \frac{1}{16\pi G}\int d\tau d^2x\sqrt{g_\infty}[-2K + 8 + 2\sigma^2], \quad (15)$$

where $K = g_\infty^{\hat{\mu}\nu}\nabla_{\hat{\mu}} n_\nu$ is the trace of the extrinsic curvature. For our class of solutions we find

$$I_{\rm ct} = \frac{\Delta\tau \mathrm{vol}_2}{16\pi G}\lim_{r\to\infty} e^{-\beta/2}[-r^2 g' + r^2 g\beta' - 4gr$$
$$+ r^2 g^{1/2}(8 + 2\sigma^2)]. \quad (16)$$

Notice that under a variation of the action $I_{\rm tot}$ with respect to β, g, $\hat{\phi}$ yields the equations of motion together with surface terms. For an on-shell variation the only terms remaining are these surface terms, and after substituting the asymptotic boundary expansion (11) (higher order terms are also required) we find

$$[\delta I_{\rm tot}]_{\rm OS} = \frac{\Delta\tau \mathrm{vol}_2}{16\pi} e^{-\beta_a/2}\bigg[\left(-\frac{1}{2}m + \frac{1}{2}\hat{\mu}\,\hat{q}\right)\delta\beta_a$$
$$- \hat{q}\delta\hat{\mu} - 4\sigma_2\delta\sigma_1\bigg]. \quad (17)$$

Note that we are keeping $\Delta\tau$ fixed in this variation. Hence we see that $I_{\rm tot}$ is stationary for fixed temperature and chemical potential (i.e., $\delta\beta_a = \delta\hat{\mu} = 0$) and for either $\sigma_2 = 0$ or fixed σ_1.

We also find that the on-shell total action is given by

$$[I_{\rm tot}]_{\rm OS} = \frac{\mathrm{vol}_2}{T}[m - \hat{\mu}\,\hat{q} - Ts] = \frac{\mathrm{vol}_2}{T}\left[-\frac{1}{2}m - 2\sigma_1\sigma_2\right], \quad (18)$$

where $s = \frac{r_+^2}{4G}$ is the entropy density of the solution and m is the energy density. The two forms of the on-shell action come from the two ways of writing the action as a total derivative given above. We note that the equality of these expressions imply a Smarr-like relation. Also note that after using $\delta\beta_a = 2\delta T/T$ (since $\Delta\tau$ is held fixed) the equality of (17) and the variation of the first form of the on-shell action in (18) imply a first law,

$$\delta m = T\delta s + \hat{\mu}\delta\hat{q} - 4\sigma_2\delta\sigma_1. \quad (19)$$

Both this Smarr relation and the first law were used to confirm the accuracy of our numerical solutions below.

For simplicity we restrict discussion to solutions with boundary condition $\sigma_1 = 0$ [12] and we interpret $TI_{\rm tot} = (\mathrm{vol}_2)(-m/2)$ as a thermodynamic potential, $\Omega(T, \mu)$. Note also that σ_2 then determines the vacuum expectation value of the operator dual to χ. Recall from [5] that writing $U = -u + v/3$, $V = 6u + v/3$, the fields u, v are dual to operators $\mathcal{O}_{u,v}$ with dimensions $\Delta_u = 4$, $\Delta_v = 6$. The truncation (6) implies that the vacuum expectation values of these dual operators are fixed by σ_2. The asymptotic expansion of u to $o(1/r^4)$ and v to $o(1/r^6)$ gives $\langle\mathcal{O}_u\rangle \propto \sigma_2^2$ and $\langle\mathcal{O}_v\rangle \propto \mu^2\sigma_2^2$.

Numerical results.—Following [3] we solved the differential equations numerically using a shooting method. We used (12) to fix the scale $\hat{\mu} = 1$. At high temperatures the black hole solutions have no scalar hair ($\sigma_2 = 0$) and are just the solutions given in (10). At a critical temperature $T_c \sim 0.042$ a new branch of solutions with $\sigma_2 \neq 0$ appears and moreover dominates the free energy. We refer to these as the unbroken and broken phase solutions, corresponding to normal and superconducting phases, respectively. In Figs. 1–3 we have plotted some features of our solutions and compared them with the solutions of the phenomenological model considered in [3].

While the results are in agreement near the critical temperature, as expected, we see marked differences as the temperature goes to zero. We have calculated the Ricci scalar and curvature invariant $\sqrt{R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}}$ at $r = r_+$ which indicate that the solutions of [3] are becoming singular but our solutions are approaching a regular zero temperature solution, without horizon, holographically dual to a quantum critical point. Indeed as $r \to r_+$ we find $\sigma \sim 1$, $\beta \sim$ const, $\hat{\phi} \sim 0$, and $g \sim \frac{16}{3}(r^2 - r_+^3/r)$,

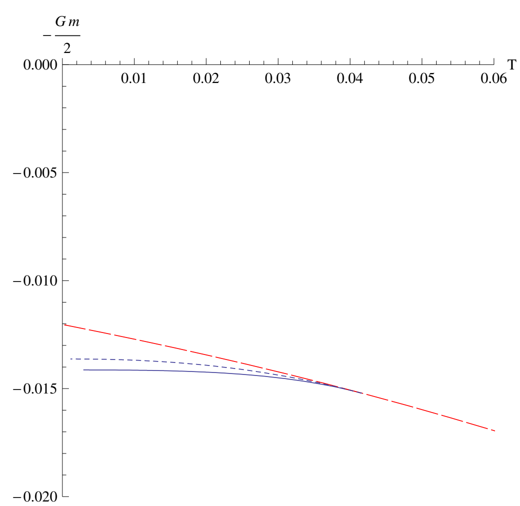

FIG. 1 (color online). Plot showing $-\frac{1}{2}Gm$ [proportional to the thermodynamic potential $\Omega(T, \mu)$] against T with fixed $\hat{\mu} = 1$, for unbroken phase solutions (long dashed red), broken phase (blue) and solutions of [3] (with their $L = 1/2$ and their $q = 2$) (dashed blue).

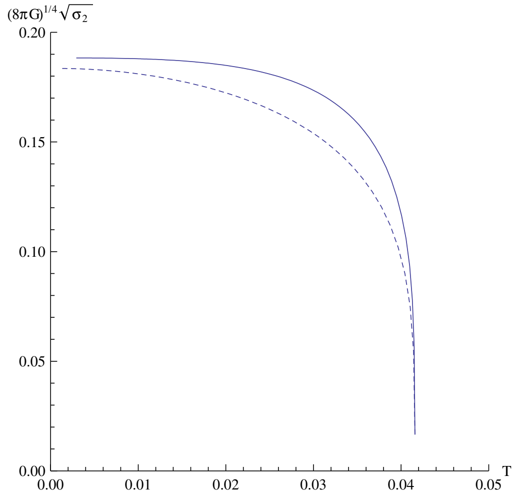

FIG. 2 (color online). Plot showing the asymptotic value of the scalar condensate, $(8\pi G)^{1/4}\sqrt{\sigma_2}$, against T (conventions as above).

and fixing $\hat{\mu} = 1$ gives $r_+ \longrightarrow 0$ in the extremal limit. In particular, the geometry near $r = r_+$ is consistent with being the exact AdS_4 solution with $\sigma = 1$, mentioned earlier, which uplifts to the $D = 11$ solution found in [11]. For such a solution $R = -64$ and $\sqrt{R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}} = 32$, agreeing with the low temperature limit seen in Figs. 1–3. The full zero temperature solution thus appears to be a charged domain wall, of the type considered in [13], connecting two AdS_4 vacua of (7), one with $\sigma = 0$ and the other with $\sigma = 1$. Interestingly this implies the entropy of the solutions vanish in the low temperature limit, unlike for the Reissner-Nordström solution (10). The asymptotic charge appears to be derived from the scalar hair, with the region near $r = r_+$ carrying no flux.

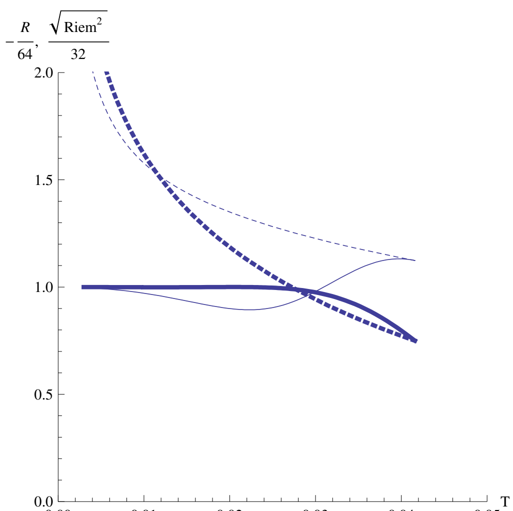

FIG. 3 (color online). Plots showing the value of the Ricci scalar (heavy lines) and $\sqrt{R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}}$ (light lines) at the horizon normalized by -64 and 32, respectively (conventions as above).

Concluding remarks.—For any seven-dimensional Sasaki-Einstein space we have constructed solutions of $D = 11$ supergravity corresponding to holographic superconductors in three spacetime dimensions. We have studied electric black holes using the action (7) whose solutions lift to $D = 11$ when $\hat{F} \wedge \hat{F} = 0$. One may consider adding magnetic charge using the full consistent truncation of [5]. Our results indicate the existence of a regular zero temperature solution which is a charged domain wall connecting two AdS_4 vacua of (7) and dual to a new quantum critical point. An important open issue is whether or not there are additional unstable charged modes for skew-whiffed $AdS_4 \times SE_7$ solutions, which condense at higher temperatures. If they exist, and dominate the free energy, then the corresponding supergravity solutions would be the appropriate ones to describe the superconductivity and not the ones that we have constructed. However, it is plausible that we have found the dominant modes for large classes of SE_7, if not all. For the specific class of deformations of the four-form that were considered in [4], it was proven that the modes that we consider are in fact the only condensing modes. It would be worthwhile extending this result to cover other bosonic and/or fermionic modes.

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Note added.—After this work was completed we received [9] which constructs solutions of string theory that are dual to superconductors in four spacetime dimensions.

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