

Percolation Transitions in Scale-Free Networks under the Achlioptas Process

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It has been recently shown that the percolation transition is discontinuous in Erdős-Rényi networks and square lattices in two dimensions under the Achlioptas process (AP). Here, we show that when the structure is highly heterogeneous as in scale-free networks, a discontinuous transition does not always occur: a continuous transition is also possible depending on the degree distribution of the scale-free network. This originates from the competition between the AP that discourages the formation of a giant component and the existence of hubs that encourages it. We also estimate the value of the characteristic degree exponent that separates the two transition types.

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The Achlioptas process (AP) is a network evolution process in which the number of vertices is fixed as N and edges are added one by one at each time step following a given rule that prevents the formation of a target pattern. Recently, Achlioptas *et al.* [1] studied the percolation transition (PT) for the Erdős-Rényi (ER) model [2] following an AP rule, called the product rule (PR) in which the formation of a giant component is discouraged. In their study, the network was developed by choosing between one of two randomly selected edges; the selected edge had a lower value of the product of the size of the two components that edge is joining. They found that the giant component emerged suddenly at a percolation threshold p_c , and that the PT was first order. This transition pattern differs drastically from the continuous PT occurring in the conventional ER model. The transition is delayed as $p_c \equiv L_c/N \approx 0.88$, larger than $p_c = 1/2$ for the conventional ER model, where L_c is the number of edges added to the system up to the transition point. More recently, Ziff [3] found the same first-order transition in the two-dimensional bond percolation clusters under AP. A similar explosive transition pattern has also been observed in a jamming transition model of Internet packets [4].

Here, we study the PT in a model scale-free (SF) network under the AP rule. SF networks contain heterogeneous degrees, and their distribution follows a power law, $P_d(k) \sim k^{-\lambda}$. To construct artificial SF networks, a stochastic model called the Chung and Lu (CL) model [5] is used. Similar to the ER model and the static model [6], the CL model starts with a fixed number of N vertices indexed $i = 1, \dots, N$. Then, a vertex i is assigned a weight of $w_i = (i + i_0 - 1)^{-\mu}$, where $\mu \in [0, 1)$ is a control parameter, and $i_0 \propto N^{1-1/2\mu}$ [7] for $1/2 < \mu < 1$ and $i_0 = 1$ for $\mu < 1/2$. Then, two different vertices (i, j) are selected with their probabilities equal to the normalized weights, $w_i/\sum_k w_k$ and $w_j/\sum_k w_k$, respectively, and an edge is added between them unless one already exists. This process is repeated until pN edges are created in the system. The obtained network is SF in degree distribution with the

exponent $\lambda = 1 + 1/\mu$. Henceforth, we will use the CL model to study the percolation transition of scale-free networks in PR (SFPR).

The mechanism by which a giant component in PT forms in conventional SF networks with $2 < \lambda < 3$ is different from that in ER networks. In an ER network, as the number of edges $L = pN$ increases in the system, multiple isolated small components are created and merged together. This process continues up to the finite percolation threshold p_c where a single giant component emerges through an abrupt coalescence of those small components. On the contrary, in SF networks with $2 < \lambda < 3$, the percolation threshold is zero in the thermodynamic limit. Thus, the giant component initially develops with the largest degree vertex as the seed, and grows continuously by aggregating small-size components. The development and growth of the giant component is the result of relatively high probability of a vertex being chosen in the giant component [8]. In the SFPR, on the other hand, two vertex pairs are selected according to the aforementioned weights. During the network growth, if two vertices get selected from the same component, an edge is created between them with no change in component size. Thus, the existence of a giant component implies that even under AP, the probability of growing the giant component is very high. This leads us to ask the following question: what is the impact of introducing the AP rule on the nature of the percolation transition in SF networks?

We obtain the following results by performing extensive numerical simulations for the SFPR model: There exists a tricritical point λ_c , estimated to be between $2.3 < \lambda_c < 2.4$, such that when $2 < \lambda \leq \lambda_c$, the transition point p_c is zero in the thermodynamic limit, and the PT is second order as in conventional SF networks. When $\lambda > \lambda_c$, however, p_c is finite, and the transition is first order. The jump in the giant component size at the first-order transition point decreases as p_c decreases. The phase diagram is depicted in Fig. 1. In finite-size systems, however, $p_c(N)$ is finite even when $\lambda < \lambda_c$ and the transition is first order.

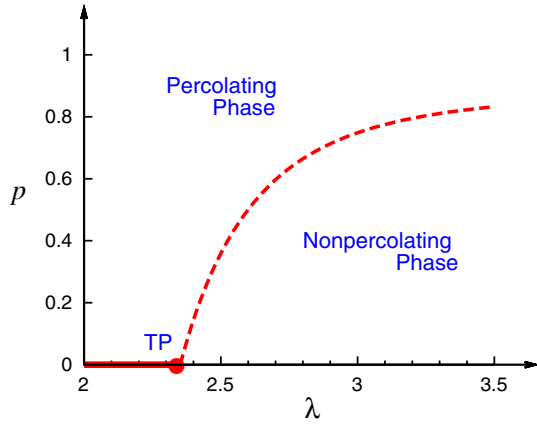


FIG. 1 (color online). Phase diagram of the percolation transition in the SFPR network. Here, $p = L/N$ is the edge density, and λ is the control parameter corresponding to the degree exponent of non-PR SF networks. A second-order (first-order) PT is represented by a solid line (dashed line). The tricritical point is denoted as “TP.”

In addition to this new feature, many other unexpected behaviors emerge.

Specifically, numerical simulations are performed for the CL model with the PR. At each time step, two candidate edges, e_1 and e_2 , are drawn from the system with respective probabilities as described previously, and added is the edge that minimizes the product of the component sizes on each end of the respective edge. Depending on the type of the edge, there are three possible cases: (i) both edges e_1 and e_2 are intercomponent ones, (ii) one edge e_1 is intracomponent, and the other e_2 intercomponent, or (iii) both edges e_1 and e_2 are intracomponent. Two subcases of (iii) are shown in (iii-a) and (iii-b) of Fig. 2. For each case, the edge added to the system is selected as follows: In (i), the edge that minimizes the product of the

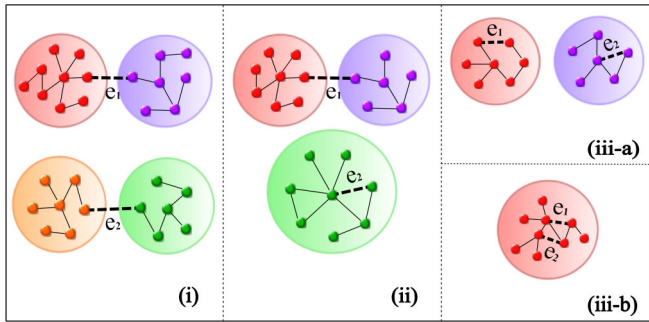


FIG. 2 (color online). A schematic diagram of the selection rules in AP for cases (i)–(iii) defined in the text. In case (i), two intercomponent edges are drawn at random, and one of them is chosen to be connected according to the product rule (PR). In case (ii), one edge is intercomponent and the other edge is intracomponent. The latter is chosen. In cases (iii-a) and (iii-b), two intracomponent edges are drawn, and one is randomly chosen to be connected.

component sizes on each side of respective edge (PR) is chosen. In (ii), the edge e_2 is chosen, leading to no change in component size. For (iii), an edge is chosen randomly between the two. A schematic picture of the selection rule in the AP is depicted in Fig. 2. Henceforth, $\lambda \equiv 1 + 1/\mu$ is a control parameter of simulation: We find that in SFPR, λ is not the resulting degree exponent, unlike in the conventional CL model (see below).

We measure the fraction of vertices in the giant component, denoted as G , averaged over $10^2 \sim 10^4$ different network configurations, as a function of p . We define the PT point, denoted by $p_c(N)$, in a system of finite size N as the point at which the local slope of G is maximal. This position is consistent with the peak position of the susceptibility defined below. We also define the discontinuity of G [9], denoted as δG , as the height of the intersection point of two tangent lines, one from the rapidly increasing transition region and the other from the smoothly increasing curve after the jump. Indeed, G shows the first-order phase transition at $p_c(N)$ in finite-size systems as shown in Fig. 3. As the parameter $\lambda \rightarrow 2$ (equivalently $\mu \rightarrow 1$), the transition point $p_c(N)$ and the jump δG decrease. To understand the behavior of $G(p)$ in the $N \rightarrow \infty$ limit, numerical simulations are performed for various system sizes in Fig. 4. We find that there exists a critical value λ_c , estimated to be between $2.3 < \lambda < 2.4$, such that for $\lambda < \lambda_c$, $p_c(N)$ decreases to zero as N increases [Fig. 4(a)] in a power-law manner $p_c(N) \sim N^{-1/\zeta}$ with $\zeta > 0$ [inset of Fig. 4(a)], and thus $p_c(N \rightarrow \infty) \rightarrow 0$. The exponent ζ depends on λ . For example, $1/\zeta \approx 0.15(1)$ for $\lambda = 2.2$. The jump δG at $p_c(N)$ decreases to zero as $\delta G \sim N^{-\beta/\zeta}$, where the exponent β also depends on λ . For example, $\beta/\zeta \approx 0.23(1)$ for $\lambda = 2.2$ [Fig. 4(b)]. Thus, we conclude that the PT is continuous in the thermodynamic limit, and Achlioptas suppression is not effective in this case. When $\lambda > \lambda_c$, however, $p_c(N \rightarrow \infty)$ converges to a finite value [inset of Fig. 4(d)]. The estimated values of $p_c(N)$ for different N s and $p_c(\infty)$ are listed in Table I. In finite-size systems, $p_c(N) - p_c(\infty) \sim N^{-1/\zeta}$. For example, estimated value of the exponent $1/\zeta = 0.29(1)$ for $\lambda = 2.8$. $\zeta \neq 1$ indicates that the first-order transition for $\lambda = 2.8$ is not criti-

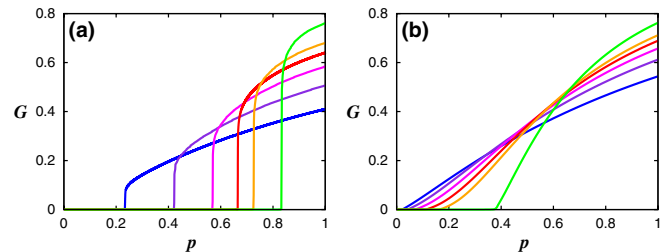


FIG. 3 (color online). The fraction G of the giant component versus the edge density p for (a) the CL model under AP, and (b) the conventional CL model. Data were obtained for networks with various control parameters λ [2.2, 2.4, 2.6, 2.8, 3.0, and 4.0 from left to right]. System size was fixed at $N = 10^7$.

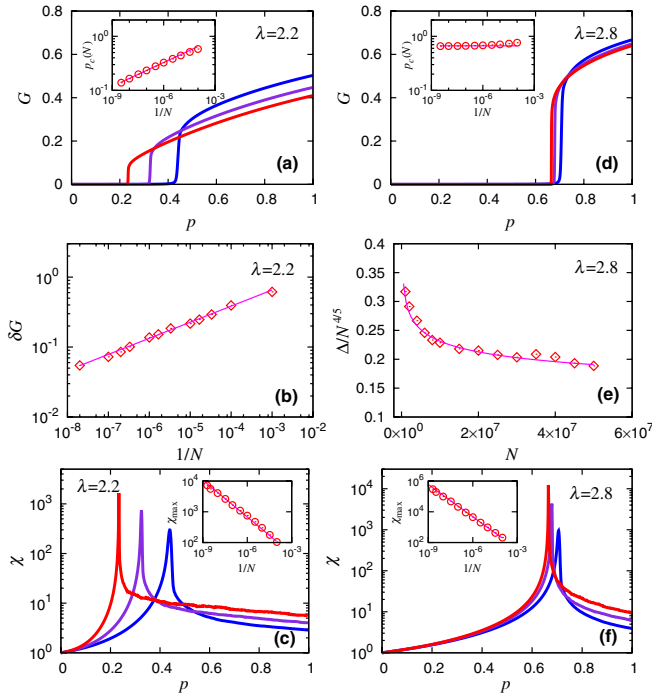


FIG. 4 (color online). (a) Same plot as Fig. 3 but for various system sizes $N = 10^5$, 10^6 , and 10^7 , from right to left. Control parameter $\lambda = 2.2$. Inset: Plot of $p_c(N)$ versus $1/N$. The solid line is a guideline with slope 0.15, indicating that $p_c(\infty) \rightarrow 0$. (b) Plot of the jump δG around $p_c(N)$ versus $1/N$. Solid line is a guideline with slope 0.23. (c) Susceptibility versus p . Inset: The peak value versus $1/N$. (d) Same as (a) for $\lambda = 2.8$. Inset: Same as the inset of (a) for $\lambda = 2.8$. Solid line is a guideline with slope 0.0, indicating that $p_c(\infty)$ is finite. (e) Scaling plot of $\Delta/N^{0.8}$ versus N for $\lambda = 2.8$, where $\Delta/N \equiv p_1 - p_0$ with p_1 and p_0 being the edge densities when the fractions of the giant component reach $G = 0.3$ and $G = N^{-1/2}$ for the first time, respectively. (f) Same as (c) for $\lambda = 2.8$. Error bars in each data point are within symbol sizes.

cal [10]. To check the nature of the PT in the thermodynamic limit, we denote $L_0 = p_0 N$ and $L_1 = p_1 N$ as the number of edges at which the value of G reaches $1/\sqrt{N}$ and 0.3, respectively. We find that there exists a scaled quantity

TABLE I. Estimated percolation threshold p_c values for finite ($N_1 = 10^6$ and $N_2 = 10^7$) and infinite system sizes, and the obtained degree exponents λ' at $p_c(\infty)$ for various λ . Errors in the last decimal points are given in parentheses.

| λ | $p_c(N_1)$ | $p_c(N_2)$ | $p_c(\infty)$ | $\lambda'(p_c)$ |
|-----------|------------|------------|---------------|-----------------|
| 2.2 | 0.33(1) | 0.23(1) | 0 | 2.8(1) |
| 2.3 | 0.42(1) | 0.33(1) | 0 | 3.0(1) |
| 2.4 | 0.49(1) | 0.42(1) | 0.18(1) | 3.1(1) |
| 2.6 | 0.60(1) | 0.57(1) | 0.52(1) | 3.5(1) |
| 2.8 | 0.68(1) | 0.66(1) | 0.65(1) | 3.8(1) |
| 3.0 | 0.73(1) | 0.73(1) | 0.72(1) | 4.2(1) |
| 4.0 | 0.83(1) | 0.83(1) | 0.83(7) | 6.3(1) |

$\Delta/N^{0.8}$ with $\Delta \equiv L_1 - L_0$, which converges to a finite value as $N \rightarrow \infty$ for $\lambda = 2.8$ [Fig. 4(e)]. The scaling factor $N^{0.8} < N$ indicates that the transition is of first order [1]. It is interesting to note that the susceptibility, defined as $\chi \equiv \sum_s s^2 n_s$ with n_s , the number of s -size components per node, and the sum excluding the largest component, diverges as $N \rightarrow \infty$ even when the transition is first order. We find that $\chi_{\max} \equiv \chi[p_c(N)] \sim N^{\gamma/\zeta}$ with $\gamma/\zeta \approx 0.4$ and 0.7 for $\lambda = 2.2$ and 2.8, respectively, shown in Figs. 4(c) and 4(f). Interestingly, $\gamma/\zeta \approx 0.7$ remains unchanged for $\lambda = 4.0$ and ∞ .

Since the second-order and the first-order transitions meet at λ_c , λ_c is a tricritical point. To estimate the position of λ_c , we measure successive slopes of the function $p_c(N)$ with respect to N for several values of λ and plot them as a function of $1/N$ in the inset of Fig. 5. We find that the successive slopes decrease to zero for $\lambda = 2.4$ and 2.5, while they converge to a finite negative value for $\lambda = 2.3$. Thus, we conclude that the tricritical point is between $2.3 < \lambda_c < 2.4$, shown in Fig. 5.

The relative frequencies of occurrence of the three cases of (i)–(iii) of Fig. 2 during the evolution is related to the degree effectiveness of AP. We find that the case (i) occurs dominantly with a probability nearly one during the period $p < p_c(N)$, in which an attached edge connects two isolated components, merging them into a larger component. Above $p_c(N)$, it decays rapidly since a giant component is already there. The cases (ii) and (iii) begin to occur when p is close to p_c . Next, we examine the component-size distribution during the evolution. In early time regime $p \ll p_c(N)$, the component-size distribution exhibits an exponential decaying behavior. As p is increased, the distribution develops a hump in a large-size region, which is made through the coalescence of small-size components, result-

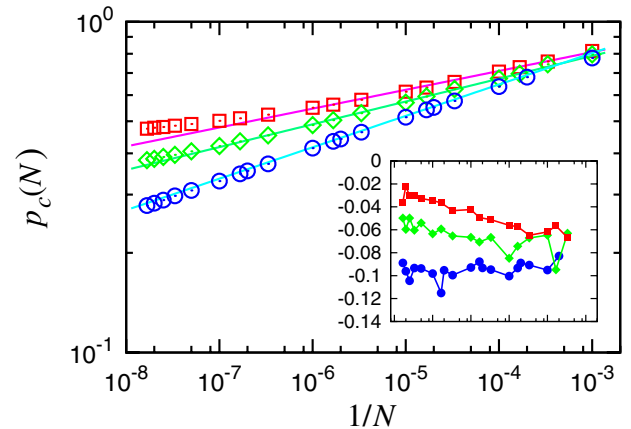


FIG. 5 (color online). Plot of $p_c(N)$ versus $1/N$ for $\lambda = 2.3$ (\circ), 2.4 (\diamond), and 2.5 (\square). Error bars in each data point are within symbol sizes. Inset: plot of successive slopes of $p_c(N)$ versus $1/N$. For $\lambda = 2.4$ (\diamond) and 2.5 (\square), the successive slopes approach zero, indicating that $p_c(\infty)$ is finite. For $\lambda = 2.3$ (\circ), the successive slopes approach a finite negative value, indicating $p_c(\infty) = 0$.

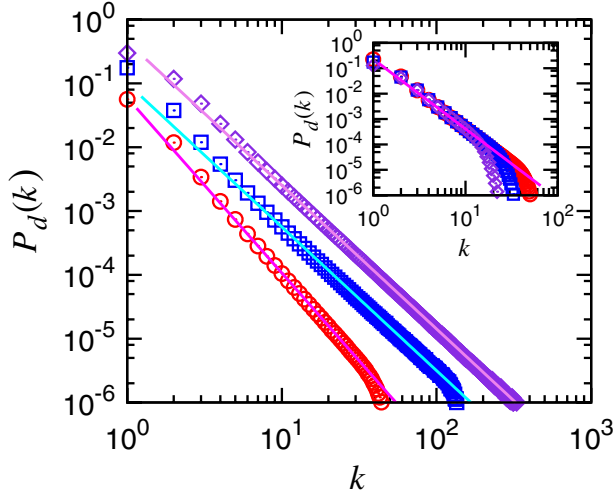


FIG. 6 (color online). Plot of the degree distribution $P_d(k)$ of the SFPR network when $\lambda = 2.2$. We find that the degree exponent $\lambda' \approx 2.8$ when $p = 0.234 \approx p_c(N)$ (\circ), and $\lambda' \approx 2.25$ when $p = 0.8$ (\diamond). The degree distribution of the conventional CL network at $p = 0.234$ is drawn (\square) for comparison. The system size is fixed at $N = 10^7$. Inset: plot of the degree distributions of the SFPR network at $p_c(N)$ for various system sizes $N = 10^5, 10^6$, and 10^7 from left to right, showing the distributions' insensitivity to system size. Data have been shifted vertically for clear view in the main panel, and for easier comparison in the inset.

ing in the abundance of large-size components. As p just passes p_c , these components finally condense into one giant component, resulting in the disappearance of the hump and a power-law distribution of component sizes. This behavior proves that the self-organization process operates during the very short transition period even under the action of AP.

Unexpectedly, the tricritical point λ_c is located in the range (2.3,2.4). To understand the underlying mechanism, we measured the degree distribution of the SFPR network. The degree distribution follows a power law $P_d(k) \sim k^{-\lambda'}$. Yet, the exponent λ' varies, depending on the edge density p for a given λ , as shown in Fig. 6. We find that λ' decreases as p increases. Numerical values of λ' obtained at $p_c(N = 10^7)$ as a function of λ are listed in Table I. Since the degree exponent λ' obtained at $p_c(N)$ turns out to be insensitive to system size N (the inset of Fig. 6), λ' at $p_c(N = 10^7)$ may be regarded as the one at $p_c(\infty)$, even though λ' is not defined at $p = 0$. Interestingly, when $\lambda < \lambda_c$, $\lambda' \leq 3$. Thus, we can assume that $p_c(\infty) = 0$ when $\lambda' \leq 3$ at p_c as long as $\lambda < \lambda_c$. This result is reminiscent of the well-known fact that $p_c(\infty) = 0$ when $\lambda \leq 3$ in conventional uncorrelated SF networks.

In summary, we have studied the percolation transition in the evolution of SF networks governed by AP. The nature of the phase transition changes from continuous to discontinuous as the degree-exponent parameter λ is tuned past a tricritical value λ_c (Fig. 1). This phenomenon originates from a competition between AP that discourages the formation of a giant component and the existence of hubs in SF networks that encourages it.

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Note added.—Shortly after the submission of this Letter, we became aware of a similar work [13] under preparation. It uses a different model from ours, the configuration model, exhibiting similar properties with some differences.

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 - [9] δG may be measured as $\delta G_2 \equiv G(p_2) - G(p_1)$, where p_1 and p_2 are defined through $dG/dp|_{p_1, p_2} = rdG/dp|_{p_c(N)}$ and $p_2 > p_1$. Here, $r (< 1)$ is a tuning parameter. We find that this alternative method does not change the N dependence of δG , regardless of r .
 - [10] The exponent ζ corresponds to $d\nu$ in the Euclidean space, where d is spatial dimension and ν is the correlation length exponent. When the first-order transition is critical, it is known that ν becomes $1/d$ and thus, $\zeta = 1$ [11,12]; however, we obtain numerically that $1/\zeta \approx 0.29, 0.69$, and 0.72 for $\lambda = 2.8, 4.0$, and ∞ , respectively. Thus, the feature of the first-order transition occurring in the current disordered system is different from the one in thermal systems.
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