Optical Hall Conductivity in Ordinary and Graphene Quantum Hall Systems

Takahiro Morimoto,[1] Yasuhiro Hatsugai,[2] and Hideo Aoki[1]
[1]*Department of Physics, University of Tokyo, Hongo, Tokyo 113-0033, Japan*
[2]*Institute of Physics, University of Tsukuba, Tsukuba, 305-8571, Japan*
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We reveal from numerical study that the optical Hall conductivity $\sigma_{xy}(\omega)$ has a characteristic feature even in the ac ($\sim$ THz) regime in that the Hall plateaus are retained both in the ordinary two-dimensional electron gas and in graphene in the quantum Hall regime, although the plateau height is no longer quantized in ac. In graphene $\sigma_{xy}(\omega)$ reflects the unusual Landau level structure. The effect remains unexpectantly robust against the significant strength of disorder, which we attribute to an effect of localization. We predict the ac quantum Hall measurements are feasible through the Faraday rotation characterized by the fine-structure constant α.

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Introduction.—There is a continuing fascination with the quantum Hall effect (QHE), despite its long history, along various avenues. While most of the works have concentrated on static properties, one direction that has not been fully explored is the optical properties of the quantum Hall system, which is exactly the purpose of the present study. One motivation comes from the fact that recent experimental advances in spectroscopy in the THz regime are making optical measurements a reality for QHE systems with the relevant energy scale being THz in magnetic fields of a few tesla [1,2]. To be more precise, these authors have observed ellipticity and Faraday rotation in a usual QHE in a two-dimensional electron gas (2DEG), and found a resonance structure at a cyclotron frequency $\omega_c \sim$ THz from the Hall angle, $\Theta_H(\omega) = \frac{1}{2}\arg(t_+(\omega)/t_-(\omega))$, which is directly connected to optical conductivity, since the transmission coefficients, $t_\pm(\omega) = 2n_0/[n_0 + n_s + \sigma_\pm(\omega)/(c\varepsilon_0)]$, are related to the optical conductivity for circularly polarized light via $\sigma_\pm(\omega) = \sigma_{xx}(\omega) \pm i\sigma_{xy}(\omega)$ [3,4].

Another motivation of the present study is the recent emergence of the physics of graphene, where the anomalous QHE specific to the ''massless Dirac'' electrons [5,6] is attracting keen interests. So the second purpose of the Letter is to study $\sigma_{xy}(\omega)$ for graphene in the QHE regime as compared with those in the ordinary 2DEG. For graphene, optical properties have begun to be studied: the longitudinal optical conductivity $\sigma_{xx}(\omega)$ has been measured through the transmission [7], or theoretically examined in terms of the cyclotron emission [8]. Here we look for features in the optical *Hall* conductivity $\sigma_{xy}(\omega)$ in graphene.

Theoretically, the question is how the static quantum Hall effect, a topological phenomenon [9,10], should evolve into the optical Hall conductivity $\sigma_{xy}(\omega)$ in the ac regime. Naively, one might expect the plateau structure in the Hall conductivity may be immediately washed out as we go into an ac regime where the topological protection no longer exists. To explore whether this intuition holds, here we have calculated the optical Hall conductivity in ordinary and graphene QHE systems to probe the ac quantum Hall physics, where the conductivity is calculated from the Kubo formula with a numerical (exact diagonalization) method, since we want to incorporate effects of Anderson localization. We start with the ordinary 2DEG, since even for 2DEG the ac conductivity has only been dealt with by a phenomenological (Drude) formalism [3] or with Maxwell's equations [11].

We shall show the following from numerical study. (i) The plateau structure in the QHE in 2DEG is retained, up to significant degree of disorder, even in the ac (THz) regime, although the heights of the plateaus are no longer quantized in the ac regime. We attribute the unexpected robustness to an effect of localization, where the existence of extended states and mobility gaps between them ensure the step structures in the ac Hall conductivity. (ii) For graphene, the optical Hall conductivity reflects the unusual Landau level structure. (iii) We then predict the ac quantum Hall effect can be detected through Faraday-rotation measurements as a step structure in $\Theta_H(\omega)$, whose magnitude is estimated to be of the order of the fine-structure constant α ($\sim$7 mrad), which is within experimental feasibility. If one utilizes a freestanding graphene, for which α has been seen as transparency [12], the rotation angle should be exactly α.

A word about the nature of the random potential here: it has long been known that the effect of the Anderson localization on the (static) conductivity is qualitatively different between short-range and long-range scatters in 2DEG [13]. Long-range potential should also be relevant to the ripples in graphene. Of particular interest, in the static graphene QHE, is that graphene has an anomalously robust $n = 0$ Landau level and the associated QHE step against disorder when it is slowly varying, which is related to topologically protected Atiyah-Singer's theorem [5,14–16]. So here we examine $\sigma_{xy}(\varepsilon_F, \omega)$ for long-range scatterers in the exact diagonalization study that takes care of the localization effects, where we have a question in mind—what will become of the Hall plateau structure in

the ac regime, especially for the $n = 0$ Landau level in graphene.

Optical Hall conductivity in ordinary QHE.—Let us first look at the optical Hall conductivity in the QHE in 2DEG as realized in GaAs/AlGaAs with a Hamiltonian, $H_0 = \frac{1}{2m^*}(\mathbf{p} + e\mathbf{A})^2$, and the current matrix elements, $j_x^{n,n'} = i\sqrt{\hbar\omega_c/2m^*}(\sqrt{n}\delta_{n-1,n'} - \sqrt{n+1}\delta_{n+1,n'})$, $j_y^{n,n'} = \sqrt{\hbar\omega_c/2m^*}(\sqrt{n}\delta_{n-1,n'} + \sqrt{n+1}\delta_{n+1,n'})$, where n, n' are the Landau indices and ω_c the cyclotron frequency.

To include the effect of disorder, we employ the exact diagonalization method for the disorder described by randomly placed scatterers with a potential $V(\mathbf{r}) = \sum_j u_j \exp(-|\mathbf{r} - \mathbf{R}_j|^2/2d^2)/(2\pi d^2)$, where d is the range of the potential, while the strength of the potential, each placed at $\mathbf{R}_j$, is assumed to take $u_j = \pm u$ with random signs so that the density of states broadens symmetrically in energy. We adopt $d = 0.7\ell$, which is comparable to the magnetic length $\ell = \sqrt{\hbar/eB}$. The degree of disorder can be characterized by $\Gamma^2/4 = u^2 N_{\text{imp}}/[2\pi(\ell^2 + d^2)L^2]$ with N_{imp} being the number of impurities, where Γ measures the Landau level broadening [8,17,18]. Diagonalization of the Hamiltonian is done by retaining 7 Landau levels. We have numerically checked that this is sufficient in the energy range considered here for the linear sample dimension of $L = 15\ell$. For the ensemble average we have taken 5000 random configurations.

To calculate the optical conductivity in the QHE system we use the Kubo formula,

$$\sigma_{xy}(\omega) = \frac{i\hbar e^2}{L^2} \sum_{\substack{\epsilon_a < \varepsilon_F \\ \epsilon_b \geq \varepsilon_F}} \frac{1}{\epsilon_b - \epsilon_a} \left(\frac{j_x^{ab} j_y^{ba}}{\epsilon_b - \epsilon_a - \hbar\omega} - \frac{j_y^{ab} j_x^{ba}}{\epsilon_b - \epsilon_a + \hbar\omega} \right), \tag{1}$$

where ϵ_a is the eigenenergy, j_x^{ab} the current matrix elements between the eigenstates, and ε_F the Fermi energy.

Figure 1 shows the results for the usual QHE system. We plot σ_{xy} on an (ω, ε_F) plane, where the $\omega = 0$ cross section corresponds to the familiar static QHE. We immediately notice two features: (i) $\sigma_{xy}(\omega)$ for a fixed value of ε_F exhibits a resonance structure around the cyclotron frequency (as observed in the experiment [1]); (ii) away from the resonance, *a step-like structure* is preserved in $\sigma_{xy}(\omega)$ as a function of ε_F for each value of ω. Although the step heights are not quantized exactly, the flatness is surprisingly preserved as seen in Fig. 1(b). If we first look at the clean limit, we can rewrite Eq. (1) as $\sigma_{xy}(\omega) \rightarrow n\frac{e^2}{h}\frac{\omega_c^2}{\omega_c^2 - \omega^2}$, since we can replace $\varepsilon_{n+1} - \varepsilon_n$ with $\hbar\omega_c$ for ε_F between n and $n+1$ Landau levels, which shows a resonance structure around $\omega \simeq \omega_c$.

The step structure is in fact a quantum effect (outside the Drude picture). In the dc QHE, the localization is the cause of the plateaus in the Aoki-Ando picture [19]. In the ac QHE, the Kubo formula, Eq. (1), contains ω, and does not simply reduce to a topological expression. In this sense the result for the robust plateaus is quite nontrivial.

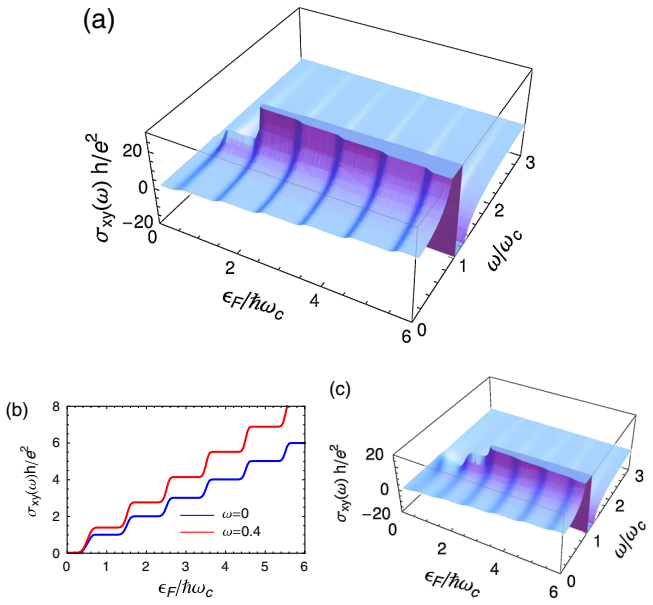

FIG. 1 (color online). Exact diagonalization result for (a) the optical Hall conductivity $\sigma_{xy}(\varepsilon_F, \omega)$ with $\Gamma = 0.2\hbar\omega_c$, (b) static [blue (dark gray)] and optical [red (light gray)] Hall conductivity $\sigma_{xy}(\varepsilon_F, \omega)$, and (c) $\sigma_{xy}(\varepsilon_F, \omega)$ with larger disorders $\Gamma = 0.5\hbar\omega_c$ in usual QHE system.

The physical insight for the unexpectedly robust ac Hall step structure is that the main contribution to the optical Hall conductivity comes from the delocalized states whose existence ensures the robust step structure in ac Hall conductivity. To be more precise, the magnitude of the current matrix elements in Eq. (1) is much larger for the extended states than for localized states, so that the optical Hall conductivity is dominated by the transitions between the extended states which reside around the center of each Landau level, while the localized states give rise to the step structure. Thus the message here is that the existence of localized and extended states manifests itself as step structures even in the ac regime.

Optical Hall conductivity in graphene.—We now turn to the optical Hall conductivity in graphene. Here we again adopt the exact diagonalization method for the disorder potential introduced by randomly placed scatterers. When the range of the random potential is much larger than the lattice constant in graphene, the scattering between K and K' points in the Brillouin zone is suppressed, so that we can assume the random term takes a diagonal form in the Dirac Hamiltonian as

$$H_0 + V = v_F \begin{pmatrix} 0 & \pi^- & 0 & 0 \\ \pi^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi^+ \\ 0 & 0 & \pi^- & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} V(\mathbf{r}). \tag{2}$$

So we adopt the Dirac model as in Refs. [14,20] to obtain

wave functions and conductivity in the presence of disorder. Retaining 9 Landau levels with system size $L = 15\ell$, we have calculated $\sigma_{xy}(\varepsilon_F, \omega)$ with the Kubo formula Eq. (1) with the current matrix for graphene [8,21].

In the result, Fig. 2, we notice several features distinct from the result for the ordinary QHE system (Fig. 1): (i) The optical Hall conductivity $\sigma_{xy}(\varepsilon_F, \omega)$ exhibits a more complex structure, which reflects the Landau levels, $\mathrm{sgn}(n)\sqrt{|n|}\hbar\omega_c$ with $\omega_c = v_F\sqrt{2eB/\hbar}$, that are not uniformly spaced for the massless Dirac dispersion. Thus a series of resonances appear around many allowed transitions, $|n| - |n'| = \pm 1$. More precisely, the low-ω resonances appear as $n \longrightarrow n + 1$ transitions, while resonances $(-n \longrightarrow n + 1)$ across the Fermi point emerge in larger ω region. (ii) Away from these resonances, we again observe that *step-like structures* remain in the optical Hall conductivity, as clearly seen in Fig. 2(b). Because of the electron-hole symmetry, $\sigma_{xy}(\varepsilon_F, \omega)$ is odd in ε_F throughout, so the step structure is symmetric as well.

If we more closely look at the result, while the ac Hall steps for larger values of $|n|$ are smeared for smaller values of ω because different Landau levels sit close to each other, the ac Hall steps for small values of $n(= 0, \pm 1)$ are robust. Specifically, the $n = 0$ step remains up to Γ as large as $0.7\hbar\omega_c$. One reason should be the $n = 0$ Landau level stands alone, but another one we note is the electron-hole symmetry. Namely, the self-energy (arising from the randomness) in Green's function contains off-diagonal elements between $\pm n$ Landau levels, where $n \neq 0$ states are significantly affected by this effect, while the $n = 0$ Landau level has itself as the electron-hole partner with no off-diagonal element. In terms of a Hamiltonian, graphene QHE is the square root of the usual QHE, so that the nth Landau level in the usual QHE bifurcates into $\pm\sqrt{n}$ Landau levels in graphene but not for $n = 0$.

We note in passing that the present result in the exact diagonalization differs from what we would have with the self-consistent Born approximation [22], where the Landau level broadening and the plateau-to-plateau transition width for different Landau indices are similar $\sim\Gamma$ [21], since localization is not considered.

Robustness of the step structure.—We finally examine how the step-like structure in the optical Hall conductivity vanishes as we further increase the degree of disorder. We have calculated $\sigma_{xy}(\varepsilon_F, \omega)$ against the strength of disorder, Γ, for each value of ω with exact diagonalization. For the ordinary QHE system in Fig. 3, we can see that the step structure remains up to $\Gamma \simeq 0.7\hbar\omega_c$ for each value of ω, both below and above the cyclotron resonance. While the density of states (not shown) broadens with a width $\sim\Gamma$, the step structure in the optical Hall conductivity $\sigma_{xy}(\omega)$ is

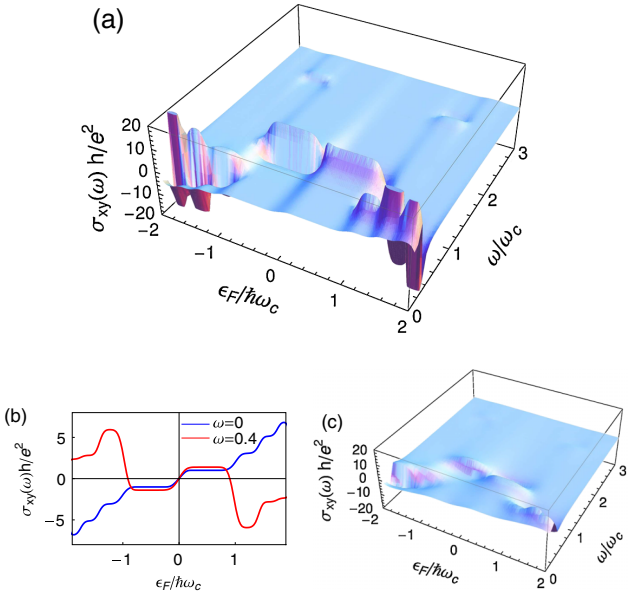

FIG. 2 (color online). Exact diagonalization result for (a) the optical Hall conductivity $\sigma_{xy}(\varepsilon_F, \omega)$ with $\Gamma = 0.2\hbar\omega_c$, (b) static [blue (dark gray)] and optical [red (light gray)] Hall conductivity $\sigma_{xy}(\varepsilon_F, \omega)$, and (c) $\sigma_{xy}(\varepsilon_F, \omega)$ with larger disorders $\Gamma = 0.5\hbar\omega_c$ for the graphene QHE system.

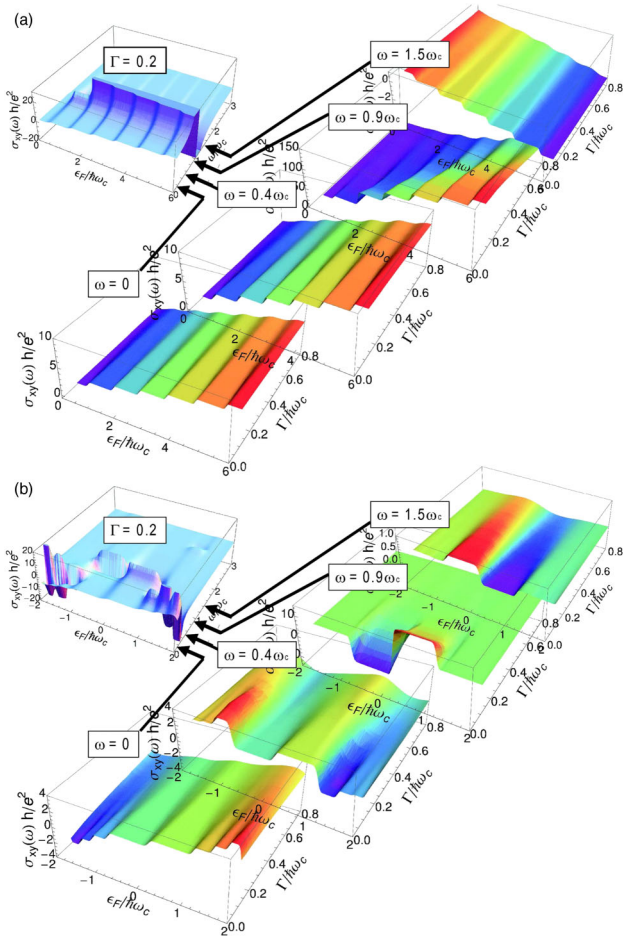

FIG. 3 (color online). Exact diagonalization result for the optical Hall conductivity, $\sigma_{xy}(\varepsilon_F, \omega)$, plotted against Fermi energy and disorder strength Γ for various values of frequency $\omega = 0, 0.4\omega_c, 0.9\omega_c, 1.5\omega_c$ in (a) the ordinary and (b) graphene QHE system.

blurred with Γ much more slowly. We can also notice in the result for $\omega = 0.9\omega_c$, very close to the cyclotron resonance where the ac Hall conductivity exceeds 100 times e^2/h, that the step structure is surprisingly preserved in such a resonant region.

Let us move on to graphene QHE (Fig. 3). The step structure for $n \neq 0$ in the optical Hall conductivity $\sigma_{xy}(\varepsilon_F, \omega)$ is less robust against disorder than in 2DEG, again due to the multiple cyclotron resonance frequencies. However, the step corresponding to $n = 0$ Landau level is robust unless ω is too close to a resonance. In the static Hall conductivity $\sigma_{xy}(\omega = 0)$, $n \neq 0$ Hall steps are smeared as soon as Landau levels are merged while the step associated with $n = 0$ Landau level is robust, which indicates that the extended states in $n = 0$ Landau level are unusually robust [14,16]. The present ac result indicates that the step in $\sigma_{xy}(\varepsilon_F, \omega)$ associated with $n = 0$ Landau level exhibits special robustness against disorder in the ac regime as well, which we take to be the effect of localization and the electron-hole symmetry.

The topological formulation of the static Hall conductivity relies on gaps between the mobility edges. The mobility gap structure is considered to bring about the robust step structure in the ac region if the frequency $\hbar\omega$ is smaller than $\hbar\omega_c$ (the energy spacing between the delocalized states). The present result (Fig. 3) does indicate that the robust step structure survives most prominently for $\omega < \omega_c$. In this sense the topological structure associated with extended states remains in the ac regime.

Faraday rotation.—To summarize, we have revealed that the optical Hall conductivity in both the ordinary QHE and graphene QHE systems has plateau structures that persist even in ac regimes for significant strengths of disorder. Finally let us mention the experimental feasibility. We propose that the ac Hall steps should be observable through accurate Faraday-rotation measurements in the THz to far-infrared spectroscopy. This is because the Faraday-rotation angle Θ_H is directly connected to the optical Hall conductivity via $\Theta_H = \frac{1}{2}\arg(\frac{t_+(\omega)}{t_-(\omega)}) \sim \frac{1}{(n_0+n_s)c\varepsilon_0}\sigma_{xy}(\omega)$, where $n_0(n_s)$ is the refractive index of air (substrate), and we have assumed $n_0 + n_s \gg \sigma_\pm/(c\varepsilon_0)$ in the last line. Hence the Faraday-rotation angle is proportional to $\sigma_{xy}(\omega)$, so that the step structure in $\sigma_{xy}(\omega)$ should be observed as jumps in Faraday-rotation measurements. We can estimate the size of the jumps $\Delta\Theta_H$ by putting $\sigma_{xy} \sim e^2/h$ (when ω is well below the resonance), so that

$$\Delta\Theta_H \sim \frac{1}{(n_0 + n_s)c\varepsilon_0}\frac{e^2}{h} \sim \frac{2}{n_0 + n_s}\alpha \sim 7 \text{ mrad}, \quad (3)$$

where $\alpha = e^2/(4\pi\varepsilon_0\hbar c)$ is the fine-structure constant. The steps in the Faraday-rotation angle should be of the order of the fine-structure constant. Recently Shimano *et al.* have achieved an experimental resolution of $\sim$1 mrad [2], so that the present effect is well within the experimental feasibility. While Nair *et al.* [12] have seen the fine-structure constant from visual transparency of graphene, the proposal here amounts to the fine-structure constant seen from a rotation.

One future problem is how we can capture $\sigma_{xy}(\omega)$ in terms of the dynamical scaling argument. There is literature on the dynamical scaling for $\sigma_{xx}(\omega)$ and associated dynamical critical exponent [23], so an extension to $\sigma_{xy}(\omega)$ should be interesting, since we have noticed that the step structure becomes slightly sharper when we go from the sample size $L = 10\ell$ to 15ℓ.

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