Spontaneous Motion of Cavity Solitons Induced by a Delayed Feedback

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We study the properties of 2D cavity solitons in a coherently driven optical resonator subjected to a delayed feedback. The delay is found to induce a spontaneous motion of a single cavity soliton that is stationary and stable otherwise. This behavior occurs when the product of the delay time and the feedback strength exceeds some critical value. We derive an analytical formula for the speed of a moving soliton. Numerical results are in good agreement with analytical predictions.

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Cavity solitons are transverse optical localized structures which belong to the class of dissipative structures found far from equilibrium [1]. Recent significant advances in their study are often due to research in nonlinear optical settings, where cavity solitons have potential applications as bits for information storage and processing. The interest in this field of research was motivated further by experimental observation of cavity solitons in driven nonlinear planar cavities [2–9]. The conditions under which localized structures and periodic patterns appear are closely related. Typically, when the modulational or Turing instability becomes subcritical, there exists a pinning domain where localized structures are stable. This subject is relatively well understood (see the latest overviews in [10]). In 1D settings, a transition from a stationary to oscillating or bouncing localized structure was observed experimentally and explained analytically for liquid crystal light valve system without delayed feedback [6]. So far, however, the investigation of the effect of the delayed feedback on the dynamics of spatially extended systems is a relatively new area of research [11]. Recently, a model for the study of cavity solitons in broad area vertical-cavity surface-emitting lasers subjected to a frequency-selective feedback was proposed in [12].

In this Letter, we study theoretically the influence of the delayed feedback on the properties of 2D cavity solitons. We consider a passive cavity filled by a two-level medium and driven by a coherent radiation beam. We focus on the regime of nascent optical bistability where the spatiotemporal dynamics can be described by the Swift-Hohenberg equation with time delay. We show that when the product of the delay time and the feedback strength exceeds some threshold, a single cavity soliton exhibits a spontaneous motion in an arbitrarily chosen direction. We derived an analytical formula for the speed of the cavity soliton. The generality of our analysis suggests that the instability leading to the spontaneous motion of cavity solitons is an universal phenomenon which does not depend on a specific type of model equation. Therefore, our conclusions should be applicable to any spatially extended system with a delay. Finally, we show that when two cavity solitons are bound together they, in addition to a forward motion, exhibit a rotation around the point corresponding to the "center of mass" of the two solitons.

Let us consider a passive cavity filled with a nonlinear media and driven coherently by an external injected field. The delayed feedback is introduced by an external reflector located at a large distance from the right facet of the cavity as shown schematically in Fig. 1. The delay time $\tau' = 2L/c$ corresponds to the round-trip time in the external cavity, where c is the speed of light and L is the optical path length. The modeling of the delayed feedback is performed in the same way as in the Rosanov [13] and Lang-Kobayashi models [14]. We assume that the laser operates in a single-longitudinal mode, the diffraction in the external cavity is fully compensated, and the feedback field is sufficiently attenuated, so that it can be modeled by a single delay term with a spatially homogeneous coefficient. Under these approximations, and close to the critical point associated with nascent bistability, the characteristic time scale is inversely proportional to the deviation from the criticality. The real distributed order parameter X that describes the deviation of the electric field envelope from its stationary value at the onset of bistability can be shown to obey the following delayed partial differential equation:

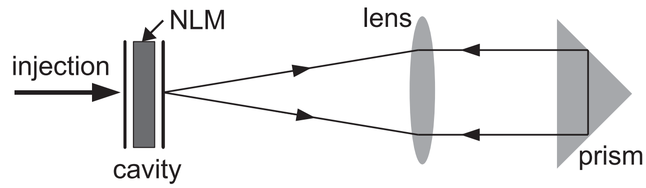

FIG. 1. Schematic setup of a passive cavity subject to delayed optical feedback. NLM—nonlinear medium. A lens in a combination with a reversing prism is used to compensate the diffraction in the external cavity.

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$$\frac{\partial X}{\partial t} = Y + CX - X^3 - 4\delta\nabla^2 X - \tfrac{4}{3}\nabla^4 X + \eta[X(t-\tau) - X], \quad (1)$$

which we shall call the delayed Swift-Hohenberg equation (DSHE). Here (x, y) are the transverse coordinates and $\nabla^2 = \partial^2_{xx} + \partial^2_{yy}$ is the transverse Laplacian. The parameter Y describes the deviation of the injection field amplitude from its critical value, C is the cooperativity parameter, τ is the normalized delay time, δ is the detuning parameter, and η is the feedback rate. Note that the product $\eta\tau$ is dimensionless and does not depend on the choice of the time scale in Eq. (1). This product is proportional to the reflectivity of the external mirror times the ratio of the external and internal cavity lengths [15]. Taking into account the phase difference between emitted and reinjected light, we obtain $\eta\tau \propto r(L/l)\cos\frac{2L}{\lambda_0}$, where λ_0 is the wavelength of the light. In Eq. (1) we subtract the cavity field from its delayed value $X(x, y, t-\tau)$, so that when we put $\eta = 0$ we recover the homogeneous steady states of the system, which are given by $Y = X_s(X_s^2 - C)$.

In the absence of the delay (i.e., at $\eta = 0$) Eq. (1) is the Swift-Hohenberg equation (SHE) [16]. It is one of the most studied partial differential equation and constitutes a paradigmatic evolution equation that exhibits not only periodic patterns but also localized structures. Stable stationary localized structures are homoclinic solutions of Eq. (1) with $\partial X/\partial t = 0$; they exist in the subcritical domain where a uniform solution and a branch of spatially periodic solution are both linearly stable [17,18]. An important property of the SHE is that it has a gradient structure, i.e., admits a potential or a Lyapunov functional, and any perturbation evolves towards a stationary homogeneous or nonhomogeneous distribution of light in the transverse plane. The existence of a Lyapunov functional pushes the time-evolution toward the state for which the functional has the smallest possible value compatible with the boundary conditions. However, in the presence of delay term $X(x, y, t-\tau)$ the DSHE equation (1) loses the gradient structure. Typical manifestation of this effect is shown in Fig. 2. When the product of the delay time and the feedback strength is small, numerical simulations of Eq. (1) with periodic boundary conditions show that localized structures are stable and stationary; i.e., $-0.98 \lesssim \eta\tau \leq 0$. However, for a sufficiently large product of the delay time and the feedback strength ($\eta\tau \lesssim -0.98$), a cavity soliton exhibits a motion with a constant velocity as shown in Fig. 2. Numerical simulations of Eq. (1) have been performed using a classical spatial finite-difference method with forward temporal Euler integration. The initial condition corresponded to the homogeneous steady state perturbed at one grid point (the magnitude of the perturbation was $\Delta X = 5$).

Let us determine the instability threshold leading to the appearance of a moving soliton solution. Consider a circularly symmetric stationary localized solution $X = u_0(\mathbf{r})$, $\mathbf{r} = (x \quad y)^T$ of the SHE; i.e., it satisfies Eq. (1) with $\eta = 0$. Since this solution is independent of time it remains a stationary localized solution of Eq. (1) for all η. Linear stability of this solution can be analyzed by substituting $X = u_0(\mathbf{r}) + \psi(\mathbf{r})e^{\lambda t + i\omega t}$ into Eq. (1) and collecting the linear in ψ terms:

$$L\psi = [\lambda + i\omega - \eta(e^{-\lambda\tau - i\omega\tau} - 1)]\psi, \quad (2)$$

where the self-adjoint linear operator $L = C - 3X_0^2 - 4\delta\nabla^2 - (4/3)\nabla^4$ describes the stability of the solution X_0 within the framework of the SHE. Notice that any eigenfunction ψ_μ of the operator L solves Eq. (2) which then transforms to

$$\mu = \lambda + i\omega - \eta(e^{-\lambda\tau - i\omega\tau} - 1), \quad (3)$$

where μ is the eigenvalue of L corresponding to the eigenfunction ψ_μ ($L\psi_\mu = \mu\psi_\mu$). Thus, the solution will be stable at a nonzero η if $\lambda(\mu)$ determined by relation (3) remains negative as μ runs the spectrum of L. Bifurcation point corresponds to $\lambda(\mu)$ vanishing at some

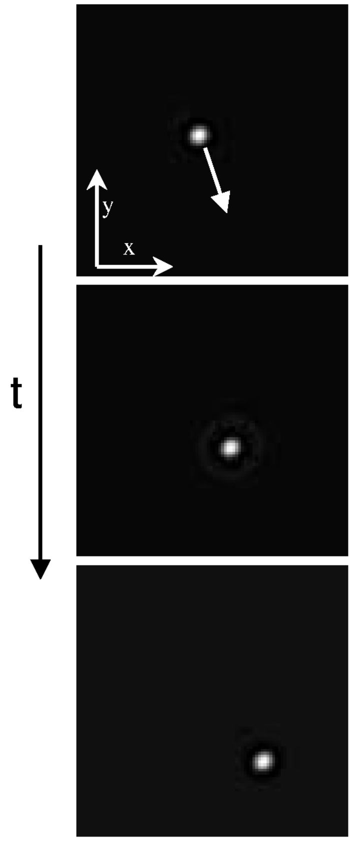

FIG. 2. Examples of 2D moving localized structures in time t obtained by numerical simulation of the model Eq. (1). with $Y = 0.25$, $C = 1$, $\delta = -0.4$, $\eta\tau = -0.98$. These parameter values correspond, e.g., to the reflectivity of the external mirror $r = 0.03$, the laser cavity length $l = 100$ μm, and the external cavity length $L \gtrsim 3$ cm [15]. Light amplitude maxima are plain white. The integration mesh is 96×96 and the time step is 0.0025.

μ. As the soliton is stable at $\eta = 0$, we have $\mu \leq 0$. Note that due to the translational symmetry operator L has two zero eigenvalues corresponding to the translational neutral modes given by the two components of the vector $(\psi_{0x}\psi_{0y})^T = \nabla u_0$; they satisfy $L\psi_{0x,0y} = 0$.

One can see from Eq. (3) that with the increase of the feedback strength $|\eta|$ the loss of stability of the solution $X = u_0(\mathbf{r})$ of Eq. (1) happens at $\eta = -1/\tau$ and this corresponds to $\mu = 0$, $\omega = 0$. At this point the eigenvalue $\lambda + i\omega = 0$ becomes fourfold degenerate with the geometrical multiplicity 2. The corresponding eigenfunctions are the neutral modes ψ_{0x} and ψ_{0y}. At the bifurcation point each of these two neutral modes corresponds to a 2×2 Jordan block.

In order to calculate analytically the soliton velocity near the bifurcation point $\eta = -1/\tau$, we assume that near this point up to the leading terms in the small velocity $v = |\mathbf{v}|$ the cavity soliton preserves its form during the motion. Thus, we make the ansatz $X(\mathbf{r}, t) = u_0(\mathbf{R})$, where $\mathbf{R} = \mathbf{r} - \mathbf{v}t$. Taylor expansion near $v = 0$ gives $u(\mathbf{R} - \mathbf{v}\tau) = u_0 + v\tau u_1 + (v\tau)^2 u_2/2 + (v\tau)^3 u_3/6 + (v\tau)^4 u_4/24 + O(v^5)$, with $u_0 = u_0(\mathbf{R})$, $u_p(\mathbf{R}) = v^{-1}(\mathbf{v} \cdot \nabla u_{p-1}(\mathbf{R}))$, $p = 1, 2, 3, 4$. By plugging this into Eq. (1), multiplying both sides by u_1, and integrating from $-\infty$ to $+\infty$ we obtain

$$-(1 + \eta\tau)\int_{-\infty}^{+\infty} u_1^2 d\mathbf{R} = -\frac{\eta v^2 \tau^3}{6}\int_{-\infty}^{+\infty} u_1 u_3 d\mathbf{R} + O(v^5) \tag{4}$$

[we used that $\int_{-\infty}^{+\infty} u_1(a_1\nabla^2 u_0 + a_2\nabla^4 u_p)d\mathbf{R} = 0$ for $p = 0, 2, 4$, which follows from the symmetry property $u_p(-\mathbf{R}) = (-1)^p u_p(\mathbf{R})$]. In addition, using integration by parts, the integral $\int_{-\infty}^{+\infty} u_1 u_3 d\mathbf{R}$ in the right-hand side of (4) can be replaced by $-\int_{-\infty}^{+\infty} u_2^2 d\mathbf{R}$. From Eq. (4) we see that when approaching the bifurcation point $v \to 0$, we should have $1 + \eta\tau = O(v^2)$. Because of the rotational symmetry of the model equation, the direction of the soliton velocity $\mathbf{v}$ is arbitrary. Therefore, the instability leading to the emergence of a moving soliton can be referred to as a circle pitchfork bifurcation [19]. Let us now introduce a small deviation from the bifurcation point, i.e., $\eta = -\tau^{-1} - \epsilon$, with $\epsilon \ll 1$. Substituting this relation into Eq. (4) and neglecting second order terms ϵ^2, we obtain

$$v = \pm Q\sqrt{\frac{6\epsilon}{\tau}} \quad \text{with} \quad Q = \sqrt{\frac{\int_{-\infty}^{+\infty} u_1^2 d\mathbf{R}}{\int_{-\infty}^{+\infty} u_2^2 d\mathbf{R}}}. \tag{5}$$

The constant Q has been evaluated numerically. For $\delta = -0.45$, and $\eta\tau = -1.025$ ($\epsilon = 0.025$), we have $Q \approx 1.33$. To compare with a direct 2D numerical simulations, we first calculate the speed given by Eq. (5) and that of the single 2D cavity soliton obtained by a direct numerical simulation. This is displayed in Fig. 3. We can see that near the bifurcation point $\eta\tau = -1$ the agreement is very good, while far from this point the numerical velocity becomes smaller than that obtained by numerical simulations. Next, we compare the threshold associated with the motion of the cavity soliton. Analytically we have $\eta_{\text{th}} = -1/\tau$ numeri-

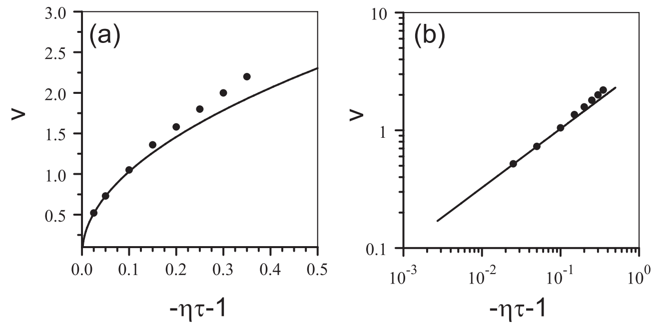

FIG. 3. Velocity of a single moving localized structures as a function of the delayed feedback strength (a). Same figure in logarithmic scale (b). Solid line indicates the velocity obtained analytically. The circles indicate the corresponding velocity obtained by numerical simulations. Same parameters as in Fig. 2.

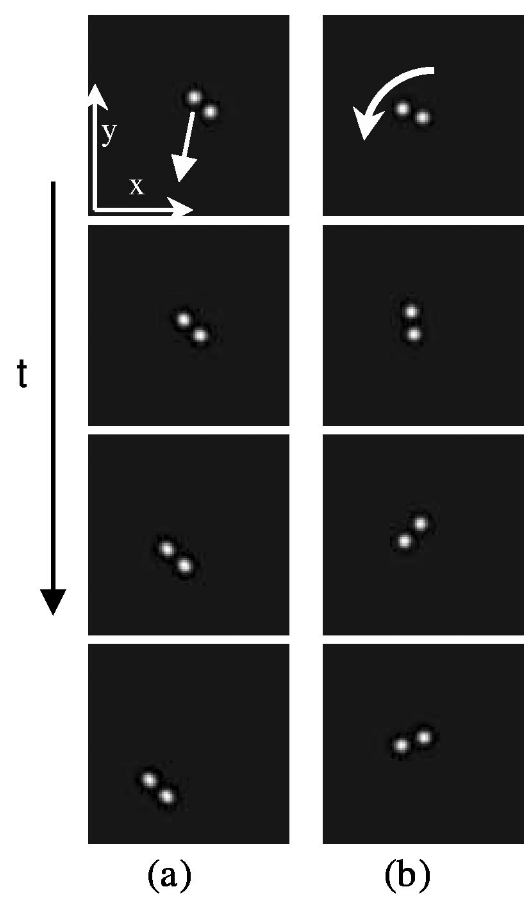

FIG. 4. Examples of 2D moving and rotating localized structures obtained by numerical simulation of Eq. (1). Parameters are the same as in Fig. 2 except (a) $\eta\tau = -0.95$ (b) $\eta\tau = -0.9$. Light amplitude maxima are plain white and the mesh integration is 96×96. The time step is 0.0025.

cally we obtain $\eta_{\text{num}} = -0.98/\tau$. We also see a good agreement between the two thresholds.

Note that the uniform motion of cavity solitons can occur in systems devoid of external delay feedback. For instance, the motion could be induced by the vorticity [20], by finite relaxation rates [21–23], by phase gradient [24], so-called Ising-Bloch transition [25,26], by the walk-off [27], or by the symmetry breaking due to off-axis feedback [28]. All these effects are absent in the model (1). In our model the presence of an external delay feedback creates a robust and controllable mechanism for the soliton motility.

When two cavity solitons are bound together, the axial symmetry of the intensity distribution is broken. As a result of this spontaneous symmetry breaking, in addition to a forward motion they exhibit a rotation as shown in Fig. 4. The analytical investigation of moving and rotating multiple-peak cavity solitons and clusters of them will be reported elsewhere.

To conclude, we have shown that the Swift-Hohenberg equation with time delay supports moving localized structures above a certain threshold which depends only on the delay parameters. More precisely, the threshold depends on the product of the delay time and the feedback strength. We have derived an analytical formula for the speed of a single-peak localized structure. The analytical results are in good agreement with numerical simulations and can be easily extended to describe similar instability in other spatially extended systems.

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