Mixtures of Bose Gases Confined in a Ring Potential

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The rotational properties of a mixture of two distinguishable Bose gases that are confined in a ring potential provide novel physical effects that we demonstrate in this study. Persistent currents are shown to be stable for a range of the population imbalance between the two components at low angular momentum. At higher values of the angular momentum, even small admixtures of a second species of atoms make the persistent currents highly fragile.

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Introduction.—One of the most fascinating phenomena associated with superfluidity [1] is the stability of persistent currents. In some remarkable experiments that have been performed recently, Bose-Einstein condensed atoms were confined in annular traps [2,3], in which persistent currents could be created and observed [4]. In an earlier experiment, the resistant-free motion of an object through a Bose-Einstein condensate below some critical velocity was also observed [5].

Motivated by these recent advances, in the present study we consider a mixture of two (distinguishable) Bose gases at zero temperature [6,7], that are confined to one dimension with periodic boundary conditions, i.e., in a ring potential, deriving a series of exact and analytic results.

The main issue of our study concerns the rotational properties of this system and the stability of persistent currents. In higher dimensions it has been argued that mixtures of Bose gases do not support persistent currents, because there is no energy cost for the system to get rid of its circulation (i.e., the line integral of the velocity field around a closed loop that encircles the ring), as long as angular momentum can be transferred between the two species [8]. Here, we demonstrate that when the total angular momentum per atom varies between zero and unity, currents are stable for a certain range of the ratio of the populations of the two species. We calculate the critical strength of the coupling for a given value of this ratio, which we determine analytically and exactly. On the other hand, for higher values of the angular momentum per atom, persistent currents in one-component systems are very fragile, as even small admixtures of a second species of atoms destabilize the currents.

Model.—Assuming a ring potential (which corresponds to a very tight annular trap along the transverse direction [9]), the Hamiltonian of the system that we study for the two components that we label as A and B is $H = H_{AA} + H_{BB} + \tilde{U}_{AB}\sum_{i=1,j=1}^{N_A,N_B}\delta(\theta_i - \theta_j)$, where

$$H_{kk} = \sum_{i=1}^{N_k} -\frac{\hbar^2}{2M_kR^2}\frac{\partial^2}{\partial\theta_i^2} + \frac{1}{2}\tilde{U}_{kk}\sum_{i\neq j=1}^{N_k}\delta(\theta_i - \theta_j), \quad (1)$$

with $k = A, B$. Here M_k are the atom masses, while $\tilde{U}_{kk} = 4\pi\hbar^2 a_{kk}/(M_kRS)$ and $\tilde{U}_{AB} = 2\pi\hbar^2 a_{AB}/(M_{AB}RS)$ are the matrix elements for zero-energy elastic atom-atom collisions (all assumed to be positive), with $M_{AB} = M_AM_B/(M_A + M_B)$ being the reduced mass. Also, R is the radius of the annulus and S its cross section, with $R \gg \sqrt{S}$.

We start from the mean-field approximation, introducing the order parameters of the two components ϕ_A and ϕ_B; later we also go beyond the mean-field approximation, diagonalizing the Hamiltonian H numerically and analytically. The resulting (coupled) nonlinear Gross-Pitaevskii-like equations are

$$-\frac{\partial^2\phi_k}{\partial\theta^2} + N_kU_{kk}|\phi_k|^2\phi_k + N_lU_{kl}|\phi_l|^2\phi_k = \mu_k\phi_k, \quad (2)$$

where $\int|\phi_k|^2d\theta = 1$. Here μ_k are the chemical potentials divided by the kinetic energy $\varepsilon = \hbar^2/(2MR^2)$, where we have assumed for simplicity equal masses for the two species, $M_A = M_B = M$. Also, $U_{kl} = \tilde{U}_{kl}/\epsilon$, with $k, l = A, B$.

Energetic stability, dynamic stability, and phase separation.—Before we turn to the rotational properties, let us consider briefly the question of phase separation. In homogeneous systems it has been shown that the condition for energetic stability of the homogeneous solution is [10–12] $U_{AB}^2 - U_{AA}U_{BB} < 0$, and also $U_{AA} > 0$, $U_{BB} > 0$. One may generalize this result for the case of a finite system, taking into account the contribution of the kinetic energy. The details of this calculation will be reported elsewhere. Here we just mention that this more general condition is $\gamma_{AB}^2 - \gamma_{AA}\gamma_{BB} < 1/4 + (\gamma_{AA} + \gamma_{BB})/2$, where we have introduced the parameters $\gamma_{k,l} = U_{k,l}\sqrt{N_kN_l}/(2\pi)$ for convenience (these parameters give the ratio between the typical interaction energy and the typical kinetic energy). As one crosses the phase boundary, the two clouds develop sinusoidal variations in their density, with an amplitude that increases continuously from zero.

The dynamic stability of the system may be examined with use of the (two coupled) Bogoliubov–de Gennes equations. Again, the details of this calculation will be

reported elsewhere. The dispersion that one obtains from this analysis is $\omega^2 = m^4 + m^2(\gamma_{AA} + \gamma_{BB} \pm \sqrt{(\gamma_{AA} - \gamma_{BB})^2 + 4\gamma_{AB}^2})$. The requirement of a real ω implies the same condition as that for energetic stability.

Effect of the periodicity on the dispersion relation.—The one-dimensional motion that we have assumed in our calculation, in combination with the periodic boundary conditions have some important consequences on the dispersion relation, which are also present in the case of a single-component gas, as shown by Bloch [13]. The matrix elements that determine the interaction energy do not depend on the quantum numbers of the angular momentum m, and also the center of mass coordinate separates from the relative coordinates. As a result, solving the problem in the interval $0 \leq l \leq 1$, where $l = (L_A + L_B)/(N_A + N_B)$ is the angular momentum per particle, then exciting the center of mass motion, we may evaluate the spectrum at any other interval $n \leq l \leq n + 1$. More specifically, if $\phi_{A,0} = \sum_m c_m \Phi_m$ and $\phi_{B,0} = \sum_m d_m \Phi_m$ are the order parameters for $0 \leq l \leq 1$, then the order parameters for $n \leq l \leq n + 1$ are given by $\phi_{A,n} = \sum_m c_m \Phi_{m+n}$, and $\phi_{B,n} = \sum_m d_m \Phi_{m+n}$.

Denoting the energy per atom for $n \leq l \leq n + 1$ as $E_n(l)/N$, then $E_n(l)/N = E_0(l_0)/N + n^2 + 2nl_0$, where $0 \leq l_0 \leq 1$, and $l = l_0 + n$. Therefore, $E_n(l)/N - l^2 = E_0(l_0)/N - l_0^2$, which are both equal to a periodic function $e(l)$, i.e., $e(l_0 + n) = e(l_0)$. Thus, we write quite generally that

$$E_n(l)/N = l^2 + e(l) = (l_0 + n)^2 + e(l_0). \quad (3)$$

In other words, the energy of the system for $n \leq l \leq n + 1$ consists of an envelope part, i.e., the first term on the right, which arises because of the center of mass excitation, plus a periodic part $e(l)$.

Furthermore, the function $e(l_0)$ is symmetric around $l_0 = 1/2$ (an example of this symmetry is demonstrated below, where it is shown that E_0/N is linear for $0 \leq l \leq x_B = 1 - x_A$ and $x_A \leq l \leq 1$). To see this, let us consider the states $\phi_A^R = \sum_m c_m \Phi_{1-m}$, and $\phi_B^R = \sum_m d_m \Phi_{1-m}$, with an l' equal to $1 - l$, or $l + l' = 1$. It turns out that the difference in the energy per particle in the states ϕ_A^R, ϕ_B^R, and ϕ_A, ϕ_B is $\Delta E/N = l' - l$. However, according to Eq. (3), $\Delta E/N = l' - l + e(l') - e(l)$, and therefore $e(l') = e(l)$, which means that $e(l_0)$ is indeed symmetric around $l_0 = 1/2$.

Rotational properties.—Since, according to what was mentioned above, the dispersion relation is quasiperiodic, in order to study the rotational properties of the gas, we restrict ourselves to the interval $0 \leq l \leq 1$. We introduce the variables $x_A = N_A/(N_A + N_B)$ and $x_B = N_B/(N_A + N_B)$, and assume without loss of generality that $x_B < x_A$, with $x_A + x_B = 1$. In what follows we also assume equal scattering lengths, and therefore $U_{AA} = U_{BB} = U_{AB} = U$. The condition of equal scattering lengths is not far from reality, with rubidium atoms in different hyperfine states being an example. Interestingly, in this case there is a series of exact, analytic results. If this condition is weakly violated, the deviations from these results will be small.

According to the result mentioned earlier, for $U_{AA} = U_{BB} = U_{AB}$ the gas is in the homogeneous phase, and it is both dynamically, as well as energetically, stable. In this case, we find that for $0 \leq l \leq x_B$ and $x_A \leq l \leq 1$, only the states with Φ_0 and Φ_1 are (macroscopically) occupied. The interaction energy of the gas is equal to that of the nonrotating system, since the total density $n(\theta) = n_A(\theta) + n_B(\theta)$ is homogeneous. As a result, the total energy of the gas varies linearly with l. These are exact results within the mean-field approximation. On the other hand, for $x_B < l < x_A$ more states contribute to the order parameters, while the dispersion relation is not linear in this interval. More specifically, let us consider the states of some fixed expectation value of the angular momentum l, $\phi_{A,0} = c_0\Phi_0 + c_1\Phi_1$, and $\phi_{B,0} = d_0\Phi_0 + d_1\Phi_1$, with $x_A|c_1|^2 + x_B|d_1|^2 = l$, and also $|c_0|^2 + |c_1|^2 = 1$, $|d_0|^2 + |d_1|^2 = 1$. The above states have a maximum value of l equal to unity. Evaluating the total energy E_0 and minimizing it, it turns out that

$$E_0/N = l + \gamma[1/2 + (x_A|c_0||c_1| - x_B|d_0||d_1|)^2], \quad (4)$$

where $N = N_A + N_B$ is the total number of atoms and $\gamma = NU/(2\pi)$. For $0 \leq l \leq x_B$ and $x_A \leq l \leq 1$, the last two terms may be set equal to each other, which means that $E_0/N = l + \gamma/2$. Remarkably, any other single-particle state cannot lower the energy and its occupancy is exactly zero. The occupancies of the single-particle states with $m = 0$ and $m = 1$ are $c_0^2 = (x_A - l)(1 - l)/[x_A(1 - 2l)]$, and $c_1^2 = l(x_B - l)/[x_A(1 - 2l)]$; d_0^2 and d_1^2 are given by similar formulas, with x_A and x_B interchanged. The same expressions hold for a mixtures of two Bose gases that are confined in harmonic traps [14], but in this case the energy is parabolic and not linear in l.

Persistent currents.—Let us now examine the question of stability of persistent currents. In the case of only one component, for $\gamma > 3/2$, the system supports persistent currents at $l = 1$ [15,16]. As we saw earlier, if one starts with $x_A = 1$ and $x_B = 0$ and increases the population of the B component, the dispersion relation is exactly linear for $x_A \leq l \leq 1$. The question is thus whether the dispersion relation has a local minimum at $l = x_A$, where we know the order parameters exactly, i.e., $\phi_{A,0} = \Phi_1$, and $\phi_{B,0} = \Phi_0$. This fact allows us to examine the region just below $l = x_A$ (and the region just above $l = x_B$, if necessary).

More specifically, if $\epsilon = x_A - l$ is a small and positive quantity, one may argue that $c_0^2 \propto c_2^2 \propto \epsilon$, while $d_{-1}^2 \propto d_1^2 \propto \epsilon$. The asymmetry between the two species arises because $c_1 = 1$ and $d_0 = 1$ at $l = x_A$. As a result, for component A, $c_0c_1^2c_2 \propto c_1^2c_2^2$, which implies that $c_2 \propto c_0$, while for component B, $d_{-1}d_0^2d_1 \propto d_{-1}^2d_0^2$, and thus $d_{-1} \propto d_1$. All the other coefficients are of higher order in ϵ, and thus negligible as $l \to x_A^-$. Since the stability of the persistent currents is determined from the slope of the dispersion

relation, we may keep only the terms which are linear in ϵ. Under these assumptions we find that the energy per particle is, up to ϵ,

$$E_0/N - \gamma/2 \approx l + 2x_A c_2^2 + 2x_B d_{-1}^2 + \gamma[x_A(c_0 + c_2) + x_B(d_{-1} + d_1)]^2, \quad (5)$$

where we have expressed c_1 in terms of c_0 and c_2, and d_0 in terms of d_{-1} and d_1 through the normalization conditions. The above expression has to be minimized under the constraint of fixed angular momentum, $l = x_A(c_1^2 + 2c_2^2) + x_B(-d_{-1}^2 + d_1^2) = x_A - \epsilon$. We do this by minimizing the function $E_0/N + \lambda[x_A(c_1^2 + 2c_2^2) + x_B(-d_{-1}^2 + d_1^2)]$, where λ is a Lagrange multiplier. The resulting equation that connects λ, x_A, x_B, and γ is $\lambda(\lambda^2 - 4)[\lambda + 2(x_B - x_A)] = 2\gamma$. For any γ, the above equation has three solutions, two of which are physically relevant. The one appears for $0 \le \lambda \le 2(x_A - x_B) = 2(2x_A - 1)$, which is ≤ 2, and the other one for $\lambda \ge 2$. The first solution gives the critical value of γ, $\gamma_{\rm cr}$, which gives a zero slope of the spectrum E_0/N for $0 \le l \le 1$, at $l = x_A^-$ as function of x_A, namely,

$$\gamma_{\rm cr} = (3/2)/(4x_A - 3). \quad (6)$$

The above expression not only gives the exact value of $\gamma_{\rm cr}$ for $x_A = 1$ and $x_B = 0$ (which is $3/2$, as mentioned earlier), but also for any (allowed) value of x_A. Since the above function diverges for $x_A \to 3/4$, persistent currents are only possible for $3/4 < x_A \le 1$.

In the intervals of higher angular momentum, $n \le l \le n + 1$ with $n \ne 0$, the situation with stability is rather different. According to Eq. (3) the periodic part of the dispersion relation $e(l)$ repeats itself in each of these intervals with a slope that is equal to $(n + 1)^2 - n^2 = 2n + 1 = 3, 5, 7, \ldots$. For $n \ne 0$ one has to use the other solution for $\lambda > 2(x_A - x_B)$. For the case of only one component, $x_A = 1$ and $x_B = 0$, this solution implies that persistent currents are stable for the values $\gamma_{\rm cr} = (2n + 1)(2n + 3)/2$, at $l = n + 1$. While the above states support persistent currents, as soon as x_B becomes nonzero—even if $x_B \to 0$ but finite—the other solution that lies in the interval $0 \le \lambda \le 2(x_A - x_B)$ has a lower energy, and destabilizes the current. In other words, the currents are very fragile with respect to admixtures of a second species of atoms. As a result, the system cannot support persistent currents at any interval other than the first one with $n \ne 0$, for $x_B \ne 0$. Figure 1 shows $\gamma_{\rm cr}$ of Eq. (6), as well as the points corresponding to $\gamma_{\rm cr} = (2n + 1)(2n + 3)/2$ for $n = 1$, 2, and 3.

To gain some physical insight on the above results, we note that for $0 \le l \le 1$, since the system is in the state $\phi_A = \Phi_1$ and $\phi_B = \Phi_0$ at $l = x_A$, it may reduce its angular momentum by either transferring some atoms of species A from Φ_1 to Φ_0, or some atoms of species B from Φ_0 to Φ_{-1}. However, the second option is energetically expensive because the angular momentum of Φ_{-1} is opposite to the angular momentum of the system. In the second interval $1 \le l \le 2$ (and in any higher one) the system is in the state $\phi_A = \Phi_2$ and $\phi_B = \Phi_1$ when $l = 1 + x_A$. In this case, however, the most efficient way for the gas to reduce its angular momentum is to transfer atoms of species B from Φ_1 to Φ_0, and not to transfer atoms of species A from Φ_2 to Φ_1, as in the first interval. It is precisely this asymmetry between the first and any other interval that allows stable persistent currents in the first interval only, but not in any other.

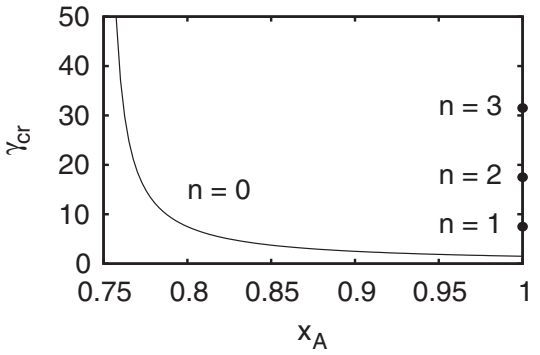

FIG. 1. The critical coupling $\gamma_{\rm cr}$ given by Eq. (6), in the interval $0 \le l \le 1$, as a function of x_A, for a ring potential. The points at $x_A = 1$ show $\gamma_{\rm cr}$ for the higher intervals of l, as explained in the text.

Beyond the mean-field approximation.—To go beyond the mean-field approximation, we have also performed numerical diagonalization of the Hamiltonian for fixed numbers of N_A, N_B, and L units of angular momentum. In the case of one component, we have confirmed the results derived within the mean-field approximation $\gamma_{\rm cr} = 3/2$ for $n = 0$, and $\gamma_{\rm cr} = 15/2$ for $n = 1$. What is even more interesting is the lowest eigenenergy of the Hamiltonian for $N_A = 17$, $N_B = 0$, as well as for $N_A = 15$, $N_B = 2$, in the range $0 \le L \le 38$, including all the single-particle states with $|m| \le 7$, for $U = \pi$, which is shown in Fig. 2 [the corresponding value of γ has to be calculated according to the formula $\gamma = (N - 1)U/(2\pi)$, which gives $\gamma = 8$]. Figure 2 indicates clearly the meta-

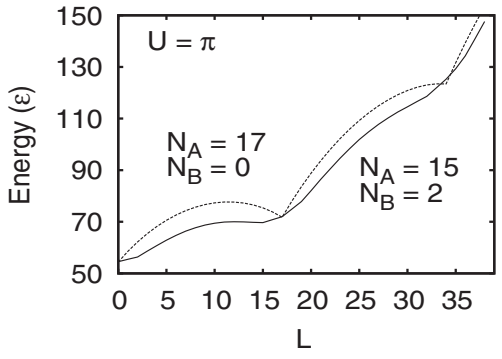

FIG. 2. The lowest eigenenergies of the Hamiltonian for $N_A = 17$, $N_B = 0$ (higher, dashed curve), as well as for $N_A = 15$, $N_B = 2$ (lower, solid curve), with $U = \pi$, in the range $0 \le L \le 38$, in the truncated space of single-particle states with $|m| \le 7$.

stability of the currents for $L = N_A$ and $L = 2N_A$ when there is only one component. With the addition of even a small second component, the local minimum around $L = 2N_A$ disappears, destroying the metastable current, while the minimum around $L = N_A$ still exists [$\gamma_{\rm cr} \approx 2.83$, according to Eq. (6)], in agreement with the mean-field approximation.

We have also found numerically that for $0 \leq L \leq N_B$ (and $N_A \leq L \leq N_A + N_B$), the (whole) excitation spectrum is given by the formula $E_q(L) = L + U/(2\pi)[q^2 + (N + 1 - 2L)q + N(N - 1)/2 - L]$, where $q = 0, 1, 2, \ldots$ in the truncated space of single-particle states with $m = 0$ and 1 (the only ones which are macroscopically occupied in the limit of large N). The lowest energy per particle $E_0(L)/N = l + \gamma/2$ agrees with the result of mean field in the limit $N \rightarrow \infty$, $L \rightarrow \infty$ with $L/N = l$ (finite) and NU finite.

A more specific case of the above spectrum may even be derived analytically with use of the Bogoliubov transformation, for $L = N_B$ (or $L = N_A$), within the same truncated space of the single-particle states with $m = 0$ and 1. Within the Bogoliubov approximation, the Hamiltonian takes the form in this case

$$H = N_B + U/(2\pi)[N(N-1)/2 + (N/2)(a_1^\dagger a_1 + b_0^\dagger b_0) + \sqrt{N_A N_B}(a_1 b_0 + a_1^\dagger b_0^\dagger)], \quad (7)$$

where a_1 is the annihilation operator of a boson of species A with angular momentum $m = 1$, and b_0 is the annihilation operator of species B with $m = 0$. This Hamiltonian is diagonalized with a Bogoliubov transformation, which implies that the eigenvalues are (assuming, for example, that $N_A > N_B$)

$$\mathcal{E}_q(N_B) = N_B + \frac{U}{2\pi}\left[\frac{N}{2}(N-2) + (N_A - N_B)(2q+1)\right]. \quad (8)$$

We then find that the difference $E_q(L = N_B) - \mathcal{E}_q(L = N_B) = Uq(q+1) \propto 1/N$, and thus vanishes for large N.

Conclusions.—This study provides an interesting illustration of the physical origin of persistent currents and, more generally, of superfluidity. The extra degrees of freedom due to the second component, combined with the assumed one-dimensionality and the periodicity of the Hamiltonian, introduce novel physical effects, which have not been known in the physics of the "traditional" superfluids.

More specifically, (i) in one-component systems, sufficiently high values of the coupling give rise to persistent currents [1]. In the present case, unless the population of the second species is sufficiently small—in which case one goes back to the one-component case—the second species provides an energetically inexpensive way for the system to get rid of its circulation: the node that is necessary to form in the component that carries the circulation, in order for the circulation to escape from the ring, is filled by the second component, very much like the coreless vortices studied in higher dimensions. (ii) The reduced dimensionality introduces another remarkable effect: while metastability of persistent currents is absent in two-component systems in higher dimensions [8,14], here the assumed one-dimensional motion makes it possible for persistent currents to be stable, at least under specific conditions. (iii) The assumed periodicity in the Hamiltonian reflects itself on the dispersion relation, which is quasiperiodic, as in the one-component problem. On the other hand, while persistent currents corresponding to the first interval of the angular momentum of the quasiperiodic part of the spectrum are stable, for higher values of the angular momentum, persistent currents are highly fragile, even for a very small admixture of a second species. This result is also in sharp contrast to the one-component case.

The results presented in our study definitely deserve experimental investigation, in order for our predictions to be confirmed. One effect that deserves both theoretical, as well as experimental, attention is the deviation from the one-dimensional motion assumed here. One may argue that as this deviation increases, competing mechanisms change the behavior of the system, interpolating between one- and two- or three-dimensional motion, thus giving rise to rich physical effects.

Last but not least, in addition to the above more theoretical remarks, the large degree of tunability of the persistent currents that we have demonstrated here also makes these systems very appealing in terms of future technological applications.

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