Transverse Angular Momentum and Geometric Spin Hall Effect of Light

Andrea Aiello,[1,*] Norbert Lindlein,[2] Christoph Marquardt,[1,2] and Gerd Leuchs[1,2]

[1]*Max Planck Institute for the Science of Light, Günter-Scharowsky-Straße 1/Bau 24, 91058 Erlangen, Germany*

[2]*Institute for Optics, Information and Photonics, University Erlangen-Nuernberg, Staudtstraße 7/B2, 91058 Erlangen, Germany*

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We present a novel fundamental phenomenon occurring when a polarized beam of light is observed from a reference frame tilted with respect to the direction of propagation of the beam. This effect has a purely geometric nature and amounts to a polarization-dependent shift or split of the beam intensity distribution evaluated as the time-averaged flux of the Poynting vector across the plane of observation. We demonstrate that such a shift is unavoidable whenever the beam possesses a nonzero transverse angular momentum. This latter result has general validity and applies to arbitrary systems such as, e.g., electronic and atomic beams.

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Introduction.—Optical angular momentum plays a key role in many fundamental and applied researches [1]. A suitably prepared beam of light may possess both a spin and an orbital angular momentum which have been traditionally associated with circular polarization and the spiraling phase front of the beam, respectively. However, it is now well known that such a distinction is not fundamental [2]. In fact, spin-to-orbital angular momentum conversion may occur in both inhomogeneous anisotropic media [3] and in tightly focused beams [4,5]. Spin-orbit coupling is also responsible for the so-called spin Hall effect of light (SHEL) [6,7] that has been recently observed in beam refraction [8] and in scattering from dielectric spheres [9]. In practice, SHEL amounts to the split of a linearly polarized beam of light into its two right-circularly and left-circularly polarized components. A similar phenomenon takes place when a light beam propagates along a curved trajectory [10]. In this case the split is uniquely determined by the trajectory geometry and can be explained in terms of geometrodynamics of the Berry phase [11].

In this Letter we present a novel fundamental effect that consists in a polarization-dependent split occurring when a beam of light is observed on a plane which is not perpendicular the propagation direction of the beam. Differently from conventional SHEL and geometrodynamics of spinning light, our split does not originate in a medium as a result of light-matter interaction, but it occurs in vacuum and is determined by the geometry of the beam-detector system only.

The structure of this Letter is as follows. We first introduce briefly the concepts of linear and angular momenta of an arbitrary beam of light. Next, we let the beam impinge on the plane $z = 0$ representing the surface of a detector, centered at the point O. Further, we assume that the axis of the beam is tilted by an angle θ with respect to the z axis (see Fig. 1). We demonstrate that the spatial distribution of the intensity of the beam $I(x, y)$ at the detector surface, evaluated as the time-averaged flux of the Poynting vector through the plane $z = 0$, depends upon the polarization of the beam. Specifically, if the beam is circularly polarized with helicity $\sigma = \pm 1$, the centroid of $I(x, y)$ [12] will be displaced along the y axis with respect to the detector center O by a distance $\delta = \lambda/(4\pi)\sigma \tan\theta$ whose magnitude is of the order of the wavelength λ of the beam. Conversely, if the beam is linearly polarized ($\sigma = 0$), then $\delta = 0$ as the beam is symmetrically split into its right-circularly and left-circularly polarized components. We describe such a "geometric SHEL" as a consequence of the tilting of the beam which is equivalent, in the detector frame, to a spin-orbit interaction that generates transverse components of the angular momentum of the beam. Finally, we discuss our results and draw some conclusions.

Linear and angular momentum of a light beam.—A monochromatic electromagnetic beam *in vacuo* possesses

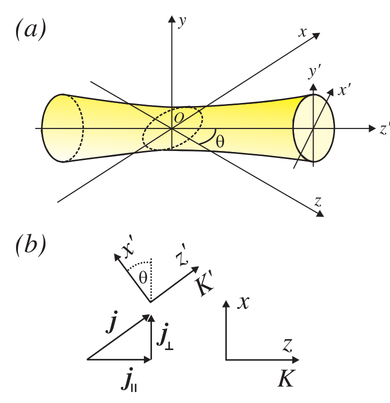

FIG. 1 (color online). (a) Geometry of the problem. (b) Components of the angular momentum density $\boldsymbol{j}$ in the beam and in the laboratory frame K' and K, respectively. Note that the beam cross section on the observation plane $z = 0$ is stretched along the x direction while the geometric SHEL shift occurs along the y axis.

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a time-averaged linear ($\boldsymbol{p}$) and an angular ($\boldsymbol{j}$) momentum density equal to

$$\boldsymbol{p}(\mathbf{r}) = \frac{\epsilon_0}{2}\,\mathrm{Re}[\mathbf{E}(\mathbf{r}) \times \mathbf{B}^*(\mathbf{r})], \tag{1a}$$

$$\boldsymbol{j}(\mathbf{r}) = \mathbf{r} \times \boldsymbol{p}(\mathbf{r}), \tag{1b}$$

where $\mathrm{Re}[\mathbf{E}(\mathbf{r})e^{-i\omega t}]$ and $\mathrm{Re}[\mathbf{B}(\mathbf{r})e^{-i\omega t}]$ are the time-harmonic electric and magnetic fields of the beam, respectively [13], and $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ is the position vector. The linear-momentum density $\boldsymbol{p}$ is equal to $1/c^2$ the Poynting vector. At any plane $z = \text{const}$, the intensity of the beam $I(\mathbf{r}) = c^2 p_z(\mathbf{r})$ can be regarded as the spatial distribution of the transverse coordinate vector $\mathbf{r}_\perp = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$. The mean value $\langle \mathbf{r}_\perp \rangle = \hat{\mathbf{x}}\langle x \rangle + \hat{\mathbf{y}}\langle y \rangle$ with respect to the distribution $I(\mathbf{r})$ is

$$\langle \mathbf{r}_\perp \rangle = \iint \mathbf{r}_\perp p_z(\mathbf{r})dxdy \Big/ \iint p_z(\mathbf{r})dxdy, \tag{2}$$

and it determines the centroid (or barycenter) of the beam at the plane $z = \text{const}$, [14]. If with $\boldsymbol{P}$ and $\boldsymbol{J}$ we denote the time-averaged linear and angular momentum of the beam per unit length [15,16] obtained by integrating $\boldsymbol{p}$ and $\boldsymbol{j}$ over the whole x-y plane at fixed z:

$$\boldsymbol{P} = \iint \boldsymbol{p}(\mathbf{r})dxdy, \qquad \boldsymbol{J} = \iint \boldsymbol{j}(\mathbf{r})dxdy, \tag{3}$$

then the denominator of Eq. (2) is, by definition, equal to P_z. From this fact and Eq. (1b), it immediately follows that

$$J_x = \langle y \rangle P_z - zP_y, \tag{4a}$$

$$J_y = zP_x - \langle x \rangle P_z, \tag{4b}$$

that, in the plane $z = 0$, reduce to

$$J_x/P_z = \langle y \rangle, \qquad J_y/P_z = -\langle x \rangle. \tag{5}$$

This remarkably simple result shows that the centroid of a beam with a nonzero transverse angular momentum per unit length $\boldsymbol{J}_\perp = \hat{\mathbf{x}}J_x + \hat{\mathbf{y}}J_y$ is displaced with respect to the center of the plane of observation $z = 0$, in a direction orthogonal to $\boldsymbol{J}_\perp$, namely, $\langle \mathbf{r}_\perp \rangle \cdot \boldsymbol{J}_\perp|_{z=0} = 0$. Moreover, Eq. (5) automatically furnishes a simple recipe to actually measure the transverse angular momentum of the beam: This can be achieved by measuring the position of its centroid at $z = 0$. With the current state-of-the-art metrology, such measurement can be performed up to and beyond the standard quantum limit [17–19]. It is worth noting that Eqs. (4) and (5) are perfectly general and their validity is not limited to electromagnetic fields, as they derive from Eq. (1b) only [20].

Now, we will calculate the time-averaged linear and angular momentum per unit length $\boldsymbol{P}$ and $\boldsymbol{J}$ of a monochromatic beam of light of wavelength $\lambda = 2\pi/k$ and frequency $\omega = kc$ propagating forward along the z axis of a Cartesian reference frame. Let

$$\mathbf{E}(\mathbf{r}, t) = E_0 \iint \boldsymbol{\mathcal{E}}(p, q)\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)dpdq \tag{6}$$

denote the time-harmonic electric field of the beam in the angular spectrum representation [13], where $\mathbf{k} = k(p\hat{\mathbf{x}} + q\hat{\mathbf{y}} + m\hat{\mathbf{z}})$ is the wave vector of the plane wave $\exp(i\mathbf{k}\cdot\mathbf{r})$ with $m = (1 - p^2 - q^2)^{1/2}$, and E_0 is the real amplitude of the electric field. Here we neglect the contribution from evanescent waves by assuming $\boldsymbol{\mathcal{E}}(p, q) \sim 0$ outside the homogenous-wave domain $\mathcal{H}$: $p^2 + q^2 \leq 1$. This condition is always satisfied by collimated beams. From Eqs. (1) and (6), it follows that

$$\boldsymbol{P}/P_0 = \iint \hat{\mathbf{k}}|\boldsymbol{\mathcal{E}}(p, q)|^2 dpdq, \tag{7}$$

and

$$\boldsymbol{J}/P_0 = \lambda\!\!\!^{-} \iint \sum_{l=1}^{3} \mathcal{E}_l^*(-i\mathbf{k} \times \boldsymbol{\nabla}_{\mathbf{k}}^{\perp})\mathcal{E}_l dpdq - \frac{i\lambda\!\!\!^{-}}{2}\iint \Big[\boldsymbol{\mathcal{E}}^* \times \boldsymbol{\mathcal{E}} + \hat{\mathbf{z}}\frac{\hat{\mathbf{k}}}{m}\cdot(\boldsymbol{\mathcal{E}}^* \times \boldsymbol{\mathcal{E}})\Big]dpdq, \tag{8}$$

where $\lambda\!\!\!^{-} = 1/k$, $k\boldsymbol{\nabla}_{\mathbf{k}}^{\perp} = \hat{\mathbf{x}}\partial/\partial q + \hat{\mathbf{y}}\partial/\partial p$, and both $\boldsymbol{P}$ and $\boldsymbol{J}$ are expressed in units of linear momentum per unit length $P_0 = \varepsilon_0 E_0^2 \lambda^2/c$. Equation (7) shows that $\boldsymbol{P}/P_0$ is just the mean value of the unit wave vector $\hat{\mathbf{k}} = \mathbf{k}/k$ with respect to the energy-density distribution in momentum space $|\boldsymbol{\mathcal{E}}(p, q)|^2$. The separation between the orbital and the spin part of the total angular momentum per unit length of the beam is displayed in the first and second rows of Eq. (8), respectively.

An interesting consequence of Eqs. (7) and (8) is that both $\boldsymbol{P}$ and $\boldsymbol{J}$ are independent from the z coordinate. However, since Eqs. (4) have general validity they must be satisfied for all values of z. Thus, by taking the derivative with respect to z of the right side of both Eqs. (4) and equating to zero, we obtain $d\langle \mathbf{r}_\perp \rangle/dz = \boldsymbol{P}_\perp/P_z = \text{const}$, which simply shows that the centroid of the beam propagates along the axis z obeying the laws of geometrical optics.

Tilted beams.—Consider a circularly polarized beam of light that propagates along the axis z' tilted by an angle θ with respect to the reference axis z, as shown in Fig. 1. In the Cartesian frame K' attached to the beam, there is a unit of spin angular momentum density directed along z': $j_{z'} \propto \sigma$ and $\boldsymbol{j}_{\perp'} = 0$, where $\sigma = \pm 1$. However, in the Cartesian frame K attached to the reference axis z, the angular momentum density of the beam will have both a longitudinal and a transverse component equal to $j_z \propto \sigma\cos\theta$ and $j_x \propto -\sigma\sin\theta$, respectively. As the cross section of the beam when seen from K is augmented by a factor $1/\cos\theta$, we expect that the transverse angular momentum per unit length of the tilted beam will go like $\boldsymbol{J}_\perp \propto \hat{\mathbf{x}}\sigma\tan\theta$. As innocent as it seems, this conclusion has a striking consequence. In fact, from Eq. (5) it immediately follows that for our tilted beam $\langle y \rangle|_{z=0} \propto \sigma\tan\theta$, which is equal either to zero or to $\pm\tan\theta$ when the beam is either linearly or circularly polarized. This means that the position of the barycenter of a tilted beam changes according to the polarization of the beam itself, and that a linearly polarized beam splits into its two left- and right-circularly polarized components. It is worth noting that since the two noncollinear axes z and z' define uniquely a "plane of incidence," then the barycenter of the tilted beam is shifted in a direction orthogonal to such plane of incidence. In this respect, this effect has the flavor of the spin Hall effect of light. However, here the split or shift has a purely geometric origin being generated by the existence, in the detector frame, of a nonzero transverse angular momentum $\boldsymbol{J}_\perp$. This is our main result that will be illustrated in the remainder of this Letter by considering the representative example of a tilted fundamental Gaussian beam.

Let us parametrize the axis of propagation $\hat{\mathbf{z}}'$ of the beam as $\hat{\mathbf{z}}' = \hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta$, and switch to dimensionless coordinates $\xi' = x'/w_0$, $\eta' = y'/w_0$, $\zeta' = z'/L$, where $L = kw_0^2/2$ and w_0 are, respectively, the Rayleigh range and the waist of the beam [13]. Then, let $\psi(\xi', \eta', \zeta') = \exp[-(\xi'^2 + \eta'^2)/(1 + i\zeta')]/(1 + i\zeta')$ denote the fundamental solution of the scalar paraxial wave equation in the beam frame. The electric and magnetic vector fields are expressible in terms of ψ as [21]

$$\mathbf{E} \propto e^{i(2\zeta'/\theta_0^2)}[\hat{\mathbf{x}}'\alpha + \hat{\mathbf{y}}'\beta + i\theta_0\hat{\mathbf{z}}'(\alpha\partial_{\xi'} + \beta\partial_{\eta'})/2]\psi, \tag{9}$$

$$\mathbf{B} \propto e^{i(2\zeta'/\theta_0^2)}[-\hat{\mathbf{x}}'\beta + \hat{\mathbf{y}}'\alpha - i\theta_0\hat{\mathbf{z}}'(\beta\partial_{\xi'} - \alpha\partial_{\eta'})/2]\psi, \tag{10}$$

where $\hat{\mathbf{u}} = \alpha\hat{\mathbf{x}}' + \beta\hat{\mathbf{y}}'$ is a complex unit vector perpendicular to z' that determines the polarization of the beam, and $\theta_0 = 2/(kw_0)$ is the angular spread of the beam [22]. From Eq. (1a), it follows that

$$\boldsymbol{p}'(\mathbf{r}') \propto |\psi|^2\left[\hat{\mathbf{z}}' + \theta_0\left(\hat{\mathbf{x}}'\frac{\xi'\zeta' - \sigma\eta'}{1 + \zeta'^2} + \hat{\mathbf{y}}'\frac{\eta'\zeta' + \sigma\xi'}{1 + \zeta'^2}\right)\right],$$

where $\sigma = i(\alpha\beta^* - \alpha^*\beta)$ denotes the helicity of the beam. Finally, we transform $\boldsymbol{p}'(\mathbf{r}')$ to the detector frame K via the map $\boldsymbol{p}'(\mathbf{r}') \to \boldsymbol{p}'(\mathbf{r}) = R(\theta)\boldsymbol{p}'(R^{-1}(\theta)\mathbf{r})$, where $R(\theta)$ is the orthogonal matrix that maps $\hat{\mathbf{z}}$ into $\hat{\mathbf{z}}'$: $\hat{\mathbf{z}}' = R(\theta)\hat{\mathbf{z}}$. The resulting expression for $\boldsymbol{p}(\mathbf{r})$ is cumbersome and it will not be reported here; the only quantity of interest is [23]

$$p_z(\xi, \eta, 0) \propto \frac{e^{-2[(\xi^2\cos^2\theta + \eta^2)/(1+\theta_0^2\xi^2\cos^2\theta)]}}{(1 + \theta_0^2\xi^2\cos^2\theta)^2}(1 + \theta_0\sigma\eta\tan\theta). \tag{11}$$

Finally, by substituting Eq. (11) into Eq. (2) and regaining unscaled coordinates $\{x, y, z\}$, we obtain

$$\langle y\rangle|_{z=0} = \lambda\!\!\!^{-}(\sigma/2)\tan\theta + O(\theta_0)^2, \tag{12}$$

in agreement with our previous qualitative argument.

As a test for Eq. (12), we have performed numerical simulations by using the program POLFOCUS [24] that simulates the intensity distribution in the focus of an arbitrary numerical aperture lens using the Debye integral [25], as long as the Fresnel number F [26] is $F \gg 1$. A set of 100×100 input plane waves and 201×1001 sampling points on the detector plane $z = 0$ were used in our simulations. For well-collimated beams ($\theta_0 = 0.01$ and $\theta_0 = 0.001$) and $\pi/2 - \theta \gtrsim \pi/180$ we found excellent agreement between analytic and numerical expressions.

The physical origin of such a shift may also be qualitatively understood with the help of Fig. 2. Let $(x', y') = (0, \mp a)$ ($a = w_0/2$) be the coordinates of the maximum and minimum of $p'_{x'}(x', y', 0)$, respectively. From Fig. 2 it may be deduced that $p'_{x'}(0, \mp a, 0) = \pm|p'_{x'}(0, a, 0)|$ and $p'_{z'}(0, a, 0) = p'_{z'}(0, -a, 0) > 0$. Furthermore, $|p'_{x'}(0, \pm a, 0) \simeq 2 \times 10^{-3}|p'_{z'}(x', a, 0)|$. Now, imagine tilting the beam by an angle θ, and look at $p_z(0, \pm a, 0)$. Rotation mixes $p'_{x'}$ and $p'_{z'}$ to produce, for sufficiently small angles θ, $p_z(0, \pm a, 0) \simeq p'_{z'}(0, a, 0)\cos\theta - p'_{x'}(0, \pm a, 0)\sin\theta$. Thus, at $y = -a$ the small term $|p'_{x'}(0, a, 0)|\sin\theta$ will be subtracted from $p'_{z'}(x', a, 0) \times \cos\theta$, while at $y = a$ it will be added. This slight imbalance of the beam intensity distribution $c^2 p_z$ causes the small shift (12) in the y direction.

Discussion and conclusions.—We have presented a quite counterintuitive and intriguing phenomenon: The position of the barycenter of the intensity distribution of a tilted beam of light varies with the polarization of the beam itself. According to the classical theory of light, we have identified the beam intensity with the flux of the Poynting vector density $\boldsymbol{s} = c^2\boldsymbol{p}$ of the beam across the detector surface [27]. However, in practice, the actually measured intensity depends on the effective response function of the detector (see, e.g., Chaps. 12 and 14 of Ref. [13]). For example, many detectors are sensitive to the electric field energy density $\propto |\mathbf{E}(\mathbf{r})|^2$ rather than s_z. In this case from Eq. (6) it follows that the y coordinate of the electric field energy-density barycenter of the beam, evaluated at $z = 0$, is

$$\langle y\rangle^{\mathrm{En}} = \lambda\!\!\!^{-}\iint i\mathcal{E}^* \cdot \frac{\partial\mathcal{E}}{\partial q}dpdq\Big/\iint|\mathcal{E}|^2 dpdq, \tag{13}$$

where the superscript "En" marks the distinction with respect to $\langle y\rangle$ defined by Eq. (2). This result should be

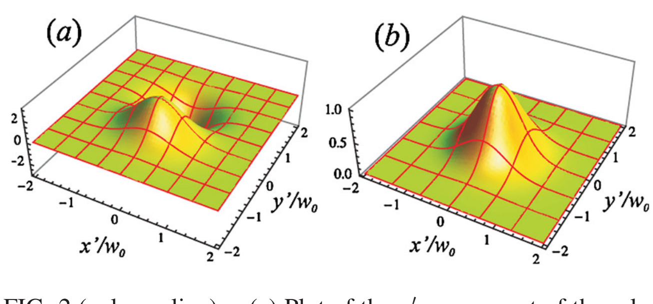

FIG. 2 (color online). (a) Plot of the x' component of the relative Poynting vector density $p'_{x'}(x', y', 0)/p'_{z'}(0, 0, 0)$ ($\times 10^3$). (b) Plot of the z' component of the relative Poynting vector density $p'_{z'}(x', y', 0)/p'_{z'}(0, 0, 0)$. In both cases the polarization is right-circular.

compared with the x component of $\boldsymbol{J}$ given by Eq. (8):

$$\frac{J_x}{P_z} = \lambda\!\!\!\bar{} \iint im\boldsymbol{\mathcal{E}}^* \cdot \frac{\partial \boldsymbol{\mathcal{E}}}{\partial q} dpdq \Big/ \iint m|\boldsymbol{\mathcal{E}}|^2 dpdq. \tag{14}$$

Comparison of Eq. (13) with Eq. (14) shows that, differently from Eq. (5), now $\langle y \rangle = J_x/P_z \neq \langle y \rangle^{\mathrm{En}}$ unless $m = (1 - p^2 - q^2)^{1/2} \approx 1$. The latter condition is satisfied by well-collimated beams with a negligible electric field component along the propagation axis z: $\mathcal{E}_z \approx 0$. If this is not the case we may have $J_x \neq 0$ and $\langle y \rangle^{\mathrm{En}} = 0$, as for our tilted beam when the incidence angle θ is bigger than the angular spread of the beam: $\theta \gg \theta_0$. It is not difficult to show that in this case $\langle y \rangle \neq 0$, but $\langle y \rangle^{\mathrm{En}} = \langle y \rangle^{\mathrm{En}}_x + \langle y \rangle^{\mathrm{En}}_y + \langle y \rangle^{\mathrm{En}}_z = 0$, since $\langle y \rangle^{\mathrm{En}}_y = 0$ and $\langle y \rangle^{\mathrm{En}}_x = -\langle y \rangle^{\mathrm{En}}_z$, where we have defined

$$\langle y \rangle^{\mathrm{En}}_l = \lambda\!\!\!\bar{} \iint i\mathcal{E}^*_l \cdot \frac{\partial \mathcal{E}_l}{\partial q} dpdq \Big/ \iint |\boldsymbol{\mathcal{E}}|^2 dpdq, \tag{15}$$

and $l \in \{x, y, z\}$. However, by using a suitable polarizing element that attenuates either $\mathcal{E}_x$ or $\mathcal{E}_z$, one can still measure a nonzero shift since, in this case, the energy imbalance causes $\langle y \rangle^{\mathrm{En}}_z \neq -\langle y \rangle^{\mathrm{En}}_x$ and $\langle y \rangle^{\mathrm{En}} \neq 0$.

In conclusion, we have demonstrated the existence of a novel optical spin-orbit effect which differs from conventional SHEL in that light-matter interaction is not required. The possible occurrence of a polarization-dependent shift in vacuum was already predicted for strongly focused beams with spherical wave fronts by Zel'dovich and coworkers [28]. However, they failed to connect such a shift with the occurrence of nonzero transverse angular momentum, and they could measure it only via a scattering process, namely, in the presence of light-matter interaction [29]. We stress again that the validity of Eq. (4), one of the main results of this Letter, is not limited to electromagnetic fields, but extends to any field satisfying Eq. (1b). With the current state-of-the-art metrology we are confident that, although tiny, geometric SHEL is detectable. In fact, experiments are in progress in our labs to confirm the existence of this intriguing phenomenon.

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*Corresponding author.
Andrea.Aiello@mpl.mpg.de

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