Sketching the Bethe-Salpeter Kernel

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An exact form is presented for the axial-vector Bethe-Salpeter equation, which is valid when the quark-gluon vertex is fully dressed. A Ward-Takahashi identity for the Bethe-Salpeter kernel is derived therefrom and solved for a class of dressed quark-gluon-vertex models. The solution provides a symmetry-preserving closed system of gap and vertex equations. The analysis can be extended to the vector equation. This enables a comparison between the responses of pseudoscalar and scalar meson masses to nonperturbatively dressing the quark-gluon vertex. The result indicates that dynamical chiral symmetry breaking enhances spin-orbit splitting in the meson spectrum.

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Understanding the spectrum of hadrons with masses less than 2 GeV is essential to revealing the essence of light-quark confinement and dynamical chiral symmetry breaking (DCSB) and describing hadrons in terms of QCD's elementary degrees of freedom. These basic questions define a frontier of hadron physics, yet there are no reliable Poincaré invariant calculations of this spectrum.

In this spectrum the $\pi(1300)$ is a radial excitation of the $\pi(140)$ [1], the $\pi(1800)$ is possibly a hybrid [2], and the dressed quarks within scalar and pesudovector mesons possess orbital angular momentum [3,4]. Hence, relative to ground-state pseudoscalar and vector mesons, these states are sensitive to different features of the light-quark interaction and to its behavior at larger distances[1,3,4]. Such systems are therefore more responsive to the dynamics of light-quark confinement. The large size of both the π-ρ mass difference and the splitting between parity partners are two consequences of DCSB, which materially influences the hadron spectrum. It is anticipated but not proven that confinement is sufficient to ensure DCSB. However, the reverse is not true [5,6].

With respect to confinement it is important to appreciate that the static potential measured in quenched lattice-regularized QCD is not related in any simple way to the question of light-quark confinement. It is a basic feature of QCD that light-quark creation and annihilation effects are nonperturbative and thus it is impossible in principle to compute a potential between two light quarks [7].

Confinement can be related to the analytic properties of QCD's Schwinger functions [6,8,9]. Hence the question of light-quark confinement can be translated into the challenge of charting the infrared behavior of QCD's *universal* β function. (This function may depend on the scheme chosen to renormalize the theory but it is unique within a given scheme [10].) Solving this well-posed problem is an elemental goal of modern hadron physics. It can be addressed in any framework enabling the nonperturbative evaluation of renormalization constants.

Through the gap and Bethe-Salpeter equations (BSEs) the pointwise behavior of the β function determines the pattern of chiral symmetry breaking. Moreover, since these and other Dyson-Schwinger equations (DSEs) [5,6,9] connect the β function to experimental observables, then the comparison between computations and observations of the hadron mass spectrum can be used to constrain the β function's long-range behavior. A nonperturbative symmetry-preserving DSE truncation is necessary to realize this goal. Steady quantitative progress can be made with a scheme that is systematically improvable [11,12]. On the other hand, one anticipates that significant qualitative advances could be made with symmetry-preserving kernel *Ansätze* that express important additional nonperturbative effects, which are difficult to capture in any finite sum of contributions. Hitherto no such *Ansatz* has been available. We remedy that.

The Poincaré covariant bound-state problem is most easily formulated for mesons. One must first solve the gap equation (f labels the quark flavor):

$$S_f(p)^{-1} = Z_2(i\gamma\cdot p + m_f^{\mathrm{bm}}) + Z_1 \int_q^\Lambda g^2 D_{\mu\nu}(p-q) \times \frac{\lambda^a}{2}\gamma_\mu S_f(q)\frac{\lambda^a}{2}\Gamma_\nu^f(q,p), \quad (1)$$

where: $D_{\mu\nu}(k)$ is the dressed-gluon propagator; $\Gamma_\nu^f(q,p)$ is the dressed-quark-gluon vertex; $\int_q^\Lambda$ is a Poincaré invariant regularization of the integral, with Λ the regularization mass scale; $m^{\mathrm{bm}}(\Lambda)$ is the Lagrangian current-quark bare mass; and $Z_{1,2}(\zeta^2,\Lambda^2)$ are, respectively, the vertex and quark wave function renormalization constants, with ζ the renormalization point—dependence upon which we do not usually make explicit. The gap equation's solution is the dressed-quark propagator, which can be written

$$S(p)^{-1} = i\gamma\cdot pA(p^2,\zeta^2) + B(p^2,\zeta^2). \quad (2)$$

The propagator is obtained from Eq. (1) augmented by a renormalization condition. A mass-independent scheme can be implemented by fixing all renormalization constants in the chiral limit [13].

Pseudoscalar and axial-vector mesons appear as poles in the inhomogeneous BSE for the axial-vector vertex, $\Gamma_{5\mu}^{fg}$. An exact form of that equation is

$$\Gamma_{5\mu}^{fg}(k;P) = Z_2\gamma_5\gamma_\mu - \int_q g^2 D_{\alpha\beta}(k-q)\frac{\lambda^a}{2}\gamma_\alpha S_f(q_+)\Gamma_{5\mu}^{fg}(q;P)S_g(q_-)\frac{\lambda^a}{2}\Gamma_\beta^g(q_-,k_-) + \int_q g^2 D_{\alpha\beta}(k-q)\frac{\lambda^a}{2}\gamma_\alpha S_f(q_+)\frac{\lambda^a}{2}\Lambda_{5\mu\beta}^{fg}(k,q;P), \quad (3)$$

where $\Lambda_{5\mu\beta}^{fg}$ is a four-point Schwinger function that is completely defined via the quark self-energy [11,12]. Owing to Poincaré covariance, one can use $q_\pm = q \pm P/2$, etc., without loss of generality. Equation (3) includes all legitimate contributions to the Bethe-Salpeter kernel and nothing extraneous. This realization generalizes the perspective of Ref. [14]. The pseudoscalar vertex, $\Gamma_5^{fg}(k;P)$, satisfies an analogous equation and has the general form

$$i\Gamma_5^{fg}(k;P) = \gamma_5[iE_5(k;P) + \gamma\cdot PF_5(k;P) + \gamma\cdot kG_5(k;P) + \sigma_{\mu\nu}k_\mu P_\nu H_5(k;P)]. \quad (4)$$

In any reliable study of light-quark hadrons the solution of Eq. (3) must satisfy the axial-vector Ward-Takahashi identity; i.e.., with Γ_5^{fg} the pseudoscalar vertex,

$$P_\mu\Gamma_{5\mu}^{fg}(k;P) = S_f^{-1}(k_+)i\gamma_5 + i\gamma_5 S_g^{-1}(k_-) - i[m_f + m_g]\Gamma_5^{fg}(k;P), \quad (5)$$

which expresses chiral symmetry and its breaking pattern. We have established that the condition

$$P_\mu\Lambda_{5\mu\beta}^{fg}(k,q;P) = \Gamma_\beta^f(q_+,k_+)i\gamma_5 + i\gamma_5\Gamma_\beta^g(q_-,k_-) - i[m_f + m_g]\Lambda_{5\beta}^{fg}(k,q;P), \quad (6)$$

where $\Lambda_{5\beta}^{fg}$ is the analogue of $\Lambda_{5\mu\beta}^{fg}$ in the pseudoscalar equation, is necessary and sufficient to ensure Eq. (5) is satisfied. Sufficiency may be verified by contracting Eq. (3) with P_μ, inserting Eq. (5), appealing to Eq. (6) and subsequently reorganizing terms so as to identify the BSE for $\Lambda_{5\beta}^{fg}$. Necessity follows from analyzing the left-hand side of Eq. (6) and using Eq. (5) to establish the right.

Consider Eq. (6). Rainbow ladder is the leading-order term in the DSE truncation of Refs. [11,12]. It corresponds to $\Gamma_\nu^f = \gamma_\nu$, in which case Eq. (6) is solved by $\Lambda_{5\mu\beta}^{fg} \equiv 0 \equiv \Lambda_{5\beta}^{fg}$. This is the solution that indeed provides the rainbow-ladder forms of Eq. (3). Such consistency will be apparent in any valid systematic term-by-term improvement of the rainbow-ladder truncation.

However, Eq. (6) is far more than merely a device for checking a truncation's consistency. For, just as the vector Ward-Takahashi identity has long been used to build *Ansätze* for the dressed-quark-photon vertex [9,15], Eq. (6) provides a way to construct a symmetry preserving kernel of the BSE that is matched to any reasonable *Ansatz* for the quark-gluon vertex that appears in Eq. (1). With this powerful capacity Eq. (6) realizes a long-standing goal.

To illustrate, suppose that in Eq. (1) one employs an *Ansatz* for the quark-gluon vertex which satisfies

$$P_\mu i\Gamma_\mu^f(k_+,k_-) = \mathcal{B}(P^2)[S_f^{-1}(k_+) - S_f^{-1}(k_-)], \quad (7)$$

with $\mathcal{B}$ flavor independent. NB. While the true quark-gluon vertex does not satisfy this identity, owing to the form of the Slavnov-Taylor identity which it does satisfy, it is plausible that a solution of Eq. (7) can provide a reasonable pointwise approximation to the true vertex.

Given Eq. (7), then Eq. (6) entails ($l = k - q$)

$$il_\beta\Lambda_{5\beta}^{fg}(k,q;P) = \mathcal{B}(l^2)[\Gamma_5^{fg}(q;P) - \Gamma_5^{fg}(k;P)], \quad (8)$$

with an analogous equation for $P_\mu l_\beta i\Lambda_{5\mu\beta}^{fg}(k,q;P)$.

This identity can be solved to obtain

$$\Lambda_{5\beta}^{fg}(k,q;P) := \mathcal{B}((k-q)^2)\gamma_5\bar{\Lambda}_\beta^{fg}(k,q;P), \quad (9)$$

with, using Eq. (4) and writing $\ell = (q+k)/2$,

$$\bar{\Lambda}_\beta^{fg}(k,q;P) = 2\ell_\beta[i\Delta_{E_5}(q,k;P) + \gamma\cdot P\Delta_{F_5}(q,k;P)] + \gamma_\beta\Sigma_{G_5}(q,k;P) + 2\ell_\beta\gamma\cdot\ell\Delta_{G_5}(q,k;P) + [\gamma_\beta,\gamma\cdot P]\Sigma_{H_5}(q,k;P) + 2\ell_\beta[\gamma\cdot\ell,\gamma\cdot P]\Delta_{H_5}(q,k;P), \quad (10)$$

with $\Sigma_\Phi(q,k;P) = [\Phi(q;P) + \Phi(k;P)]/2$ and $\Delta_\Phi(q,k;P) = [\Phi(q;P) - \Phi(k;P)]/[q^2 - k^2]$.

Now, given any *Ansatz* for the quark-gluon vertex that satisfies Eq. (7), then the pseudoscalar analogue of Eq. (3) and Eqs. (1), (9), and (10) provide a symmetry-preserving closed system whose solution predicts the properties of pseudoscalar mesons. The system can be used to anticipate, elucidate and understand the impact on hadron properties of the rich nonperturbative structure expected of the fully-dressed quark-gluon vertex in QCD.

To exemplify, we consider ground-state pseudoscalar and scalar mesons composed of equal-mass u and d quarks. The inhomogeneous BSE for the scalar vertex describes scalar mesons. It is straightforward to adapt the discussion already presented to derive the scalar-vertex analogues of, e.g., Eqs. (9) and (10). (We are aware of the effects of resonant contributions to the kernel in the scalar channel [16] but they are not pertinent herein.)

To proceed we need only specify the gap equation's kernel because the BSEs are completely defined therefrom. The kernel is typically rendered by writing

$$Z_1 g^2 D_{\rho\sigma}(t)\Gamma_\sigma(q, q+t) = \mathcal{G}(t^2) D^{\rm free}_{\rho\sigma}(t)\Gamma_\sigma(q, q+t), \quad (11)$$

wherein $D^{\rm free}_{\rho\sigma}$ is the Landau-gauge free-gauge-boson propagator, $\mathcal{G}$ is an interaction model and Γ_σ is a vertex *Ansatz*. Herein we employ the Ball-Chiu (BC) model for the dressed-quark-gluon vertex [15]:

$$i\Gamma_\mu(q, k) = i\Sigma_A(q^2, k^2)\gamma_\mu + 2\ell_\mu[i\gamma\cdot\ell\Delta_A(q^2, k^2) + \Delta_B(q^2, k^2)], \quad (12)$$

where A, B appear in Eq. (2); and a simplified form of the effective interaction in Ref. [17]:

$$\frac{\mathcal{G}(\ell^2)}{\ell^2} = \frac{4\pi^2}{\omega^6} D\ell^2 e^{-\ell^2/\omega^2}. \quad (13)$$

NB. Equation (12) does not enforce $\mathcal{B} \equiv 1$ in Eq. (7): a deviation from unity can always be absorbed into the gluon propagator. These *Ansätze* are used for illustrative simplicity, not out of necessity. The status of DSE studies of propagators and vertices can be tracked from Ref. [6].

Equation (13) delivers an ultraviolet finite model gap equation. Hence, the regularization mass scale can be removed to infinity and the renormalization constants set equal to one. For comparison we also report results obtained in the rainbow-ladder truncation; namely, with

$$\Gamma_\sigma(q, p) = \gamma_\sigma. \quad (14)$$

The active parameters in Eq. (13) are D and ω but they are not independent: a change in D is compensated by an alteration of ω [5]. For $\omega \in [0.3, 0.5]$ GeV, using Eq. (14), ground-state pseudoscalar and vector-meson observables are roughly constant if $\omega D = (0.8\ \text{GeV})^3$.

We obtain meson masses from the inhomogeneous BSEs following Secs. 3.1, 3.2 of Ref. [18], with the results presented in Table I. The quark condensate reported in that table is obtained from the trace of the chiral-limit dressed-quark propagator and the chiral-limit leptonic decay constant is determined from the relation [19]:

$$(f^0_\pi)^2 = \frac{-\langle\bar{q}q\rangle^0_\zeta}{s^0_\pi(\zeta)}, \qquad s^0_\pi(\zeta) = m_\pi \frac{dm_\pi}{dm(\zeta)}\bigg|_{\hat{m}=0}. \quad (15)$$

[Remember, the renormalization point is removed to infinity when using Eq. (13).] Both the condensate and decay constant are order parameters for DCSB. It is evident that dressing the vertex amplifies this phenomenon.

Herein, for the first time, Eq. (15) can veraciously be used for a truncation whose diagrammatic content is unknown because we have enabled a direct calculation of the current-quark-mass dependence of meson masses obtained with the Ball-Chiu vertex. That dependence is depicted in Fig. 1 and compared with the rainbow-ladder result. The m dependence of the pseudoscalar meson's mass provides numerical confirmation that the axial-vector Ward-Takahashi identity is preserved by both the rainbow-ladder truncation and our BC-consistent *Ansatz* for the Bethe-Salpeter kernel. The figure also shows that the axial-vector Ward-Takahashi identity and DCSB conspire to shield the pion's mass from material variation in response to dressing the quark-gluon vertex [5,14].

TABLE I. *Upper panel*—Selected results. Current-quark masses: *upper rows*—6.4 MeV; *next two*—6 MeV; and *lowest*—5 MeV. Notes: (i) $\langle\bar{q}q\rangle^0$, f^0_π are, respectively, the chiral-limit quark condensate and pion decay constant; and (ii) $D\omega = \frac{1}{4}$ is only slightly above the critical interaction strength for DCSB in the rainbow gap equation [13], which explains the values in Row 2. *Lower panel*—Comparison between the exact chiral-limit decay constant, Eq. (15), and two oft used estimation formulae: respectively, Eqs. (C.4) and (7.57) of Ref. [9], with percentage errors in parentheses. Experimentally, $f_\pi = 0.092$ GeV. ($\omega = 0.5$ GeV throughout; $A(0)$, dimensionless; D, GeV^2; and other entries quoted in GeV.)

Vertex	D	$A(0)$	$M(0)$	$-(\langle\bar{q}q\rangle^0)^{1/3}$	f^0_π	m_π	m_σ
Equation (14), RL	$\frac{1}{2}$	0.97	0.049	0.13	0.029	0.16	0.27
Equation (12), BC		1.1	0.28	0.26	0.11	0.14	0.56
Equation (14), RL	$\frac{2}{3}$	1.1	0.21	0.21	0.071	0.14	0.44
Equation (12), BC		1.3	0.44	0.30	0.13	0.14	0.81
Equation (14), RL	1	1.3	0.40	0.25	0.091	0.14	0.64
Equation (12), BC		1.8	0.62	0.36	0.16	0.13	1.1

Vertex	D	f^0_π	$f^0_{\pi\rm PS}$	$f^0_{\pi\rm CR}$
Equation (14), RL	$\frac{2}{3}$	0.071	0.063 (10%)	0.070 (-0.4%)
Equation (12), BC		0.13	0.10 (25%)	0.12 (12%)

Since our procedure ensures that Eq. (15) provides a true result for f^0_π, we can explore the accuracy of two formulae oft used to estimate this quantity. We find that of Ref. [20] generally provides the more reliable estimate (see Table I). The estimation formulae are more reliable in rainbow-ladder truncation because they are derived under the assumption that the bound-state analogues of F_5, G_5, H_5 in Eq. (4) are zero. The importance of these amplitudes is signalled by the magnitude of $[A(0) - 1]$ [19], which, for a given mass scale in Eq. (13), is smaller in the rainbow truncation (see Table I).

In the rainbow-ladder DSE truncation, using a kernel with realistic interaction strength, one finds [17,21–23]

$$\varepsilon^{\rm RL}_\sigma := \frac{2M(0) - m_\sigma}{2M(0)}\bigg|_{\rm RL} = (0.3 \pm 0.1). \quad (16)$$

This can be contrasted with the value obtained using our *Ansatz* for the BC-consistent Bethe-Salpeter kernel; viz.,

$$\varepsilon^{\rm BC}_\sigma \lesssim 0.1. \quad (17)$$

Plainly, significant additional repulsion is present in the BC-consistent truncation of the scalar BSE.

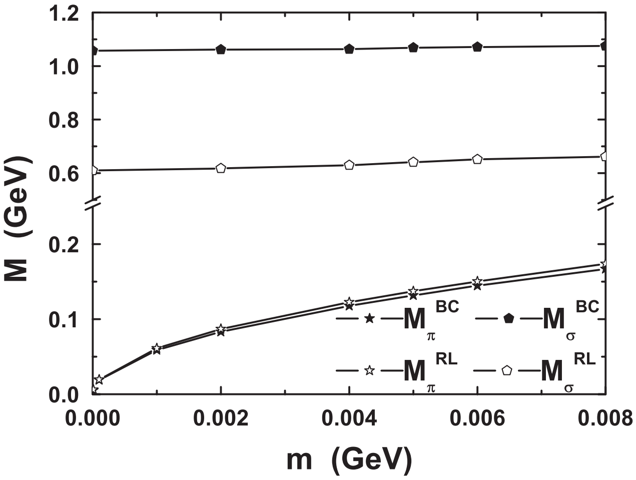

FIG. 1. Current-quark-mass dependence of pseudoscalar (lower portion) and scalar (upper) meson masses, obtained with $D = 1$ GeV^2. The Ball-Chiu vertex [BC, Eq. (12)] result is compared with the rainbow-ladder result [RL, Eq. (14)].

Scalar mesons are identified as $^3P_0$ states. This assignment reflects a constituent-quark model perspective, from which a $J^{PC} = 0^{++}$ meson must have the constituents' spins aligned and one unit of constituent orbital angular momentum. From this viewpoint a scalar is a spin and orbital excitation of a pseudoscalar meson. Extant studies of realistic corrections to the rainbow-ladder truncation show that they reduce hyperfine splitting [14]. Hence, the comparison between Eqs. (16) and (17) indicates that in a Poincaré covariant treatment the BC-consistent truncation magnifies spin-orbit splitting. We attribute this to the influence of the quark's dynamically-enhanced scalar self-energy [6] in the Bethe-Salpeter kernel.

We expect this feature to have a material impact on mesons with mass greater than 1 GeV. Indeed, *prima facie* it can plausibly overcome a long-standing shortcoming of the rainbow-ladder truncation; viz., that the splitting between vector and axial-vector mesons is too small [24]. This expectation is supported by Ref. [4] wherein, using a separable *Ansatz* for the Bethe-Salpeter kernel which depends explicitly on the strength of DCSB, a vector–axial-vector mass splitting is obtained that is commensurate with experiment.

We presented a Ward-Takahashi identity for the pseudovector Bethe-Salpeter kernel, Eq. (6), and used it to construct a symmetry-preserving *Ansatz* for this kernel, Eqs. (9) and (10), which is consistent with a large class of dressed-quark-gluon vertices whose diagrammatic content cannot be specified. Although we did not explicitly report formulae, our procedure extends readily to the vector Bethe-Salpeter equation. We were therefore able to complete the first exploration of the effect of nonperturbative vertex dressing on the masses of pseudoscalar and scalar mesons. Our results indicate that the dressed-light-quark mass function, which is inextricably connected with dynamical chiral symmetry breaking, acts to magnify spin-orbit splitting in the meson spectrum.

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[1] A. Höll, A. Krassnigg, P. Maris, C. D. Roberts, and S. V. Wright, Phys. Rev. C **71**, 065204 (2005).
[2] T. Barnes, F. E. Close, P. R. Page, and E. S. Swanson, Phys. Rev. D **55**, 4157 (1997).
[3] E. S. Ackleh, T. Barnes, and E. S. Swanson, Phys. Rev. D **54**, 6811 (1996).
[4] J. C. R. Bloch, Y. L. Kalinovsky, C. D. Roberts, and S. M. Schmidt, Phys. Rev. D **60**, 111502 (1999).
[5] C. D. Roberts, M. S. Bhagwat, A. Höll, and S. V. Wright, Eur. Phys. J. Special Topics **140**, 53 (2007).
[6] C. D. Roberts, Prog. Part. Nucl. Phys. **61**, 50 (2008).
[7] G. S. Bali, H. Neff, T. Duessel, T. Lippert, K. Schilling, and the (SESAM Collaboration), Phys. Rev. D **71**, 114513 (2005).
[8] G. Krein, C. D. Roberts, and A. G. Williams, Int. J. Mod. Phys. A **7**, 5607 (1992).
[9] C. D. Roberts and A. G. Williams, Prog. Part. Nucl. Phys. **33**, 477 (1994).
[10] W. Celmaster and R. J. Gonsalves, Phys. Rev. D **20**, 1420 (1979).
[11] H. J. Munczek, Phys. Rev. D **52**, 4736 (1995).
[12] A. Bender, C. D. Roberts, and L. von Smekal, Phys. Lett. B **380**, 7 (1996).
[13] L. Chang *et al.*, Phys. Rev. C **79**, 035209 (2009).
[14] M. S. Bhagwat, A. Höll, A. Krassnigg, C. D. Roberts, and P. C. Tandy, Phys. Rev. C **70**, 035205 (2004).
[15] J. S. Ball and T.-W. Chiu, Phys. Rev. D **22**, 2542 (1980).
[16] A. Höll, P. Maris, C. D. Roberts, and S. V. Wright, Nucl. Phys. B, Proc. Suppl. **161**, 87 (2006).
[17] P. Maris and C. D. Roberts, Phys. Rev. C **56**, 3369 (1997).
[18] M. S. Bhagwat, A. Höll, A. Krassnigg, C. D. Roberts, and S. V. Wright, Few-Body Syst. **40**, 209 (2007).
[19] P. Maris, C. D. Roberts, and P. C. Tandy, Phys. Lett. B **420**, 267 (1998).
[20] R. T. Cahill and C. D. Roberts, Phys. Rev. D **32**, 2419 (1985).
[21] P. Maris, C. D. Roberts, S. M. Schmidt, and P. C. Tandy, Phys. Rev. C **63**, 025202 (2001).
[22] L. Chang, Y.-X. Liu, M. S. Bhagwat, C. D. Roberts, and S. V. Wright, Phys. Rev. C **75**, 015201 (2007).
[23] R. Alkofer, P. Watson, and H. Weigel, Phys. Rev. D **65**, 094026 (2002).
[24] P. Maris, AIP Conf. Proc. **892**, 65 (2007).