Mass of a Spin Vortex in a Bose-Einstein Condensate

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In contrast with charge vortices, spin vortices in a two-dimensional ferromagnetic condensate move inertially (if the condensate has zero magnetization along an axis). The Magnus force, which prevents the inertial motion of the charge vortices, cancels for spin vortices, because they are composed of two oppositely rotating vortices. The inertial mass of spin vortices varies inversely with the strength of spin-dependent interactions and directly with the width of the condensate layer, and can be measured as a part of experiments on how spin vortices orbit one another. For Rb^{87} in a 1 μm thick trap, $m_v \sim 10^{-21}$ kg.

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Vortices, with their long life and concentration of energy, often provoke comparison to particles. But does their motion fit the analogy? Ordinary vortices in a fluid or superfluid do not move inertially, as particles do, because their motion is Magnus-force dominated. For example, in the absence of a force, they do not move. When a force is applied, they move perpendicular to it, a situation which is described by first-order differential equations [1]. In fact, the motion of a pair of vortices is a miniature version of how Descartes [2] explained the motion of planets, with the Sun causing the ether to whirl around, dragging the planets at the same speed. On the other hand, spinor superfluids [3–5] made out of laser-cooled atoms can have spin-current vortices. These vortices will be argued to obey Newton's laws at low speeds; in particular, they have a mass, which determines their resistance to being accelerated. The justification of these properties of spin vortices described here is inspired partly by the motion of vortex rings [6,7].

An ordinary vortex in a superconductor or superfluid can have some inertial behavior. The mass may play a role in determining the oscillation frequency of vortex lattices and the tunneling rate of vortices [8]. Observing the mass for such vortices is much more subtle than for spin vortices, though, because the Magnus force is so strong.

We will focus on the case of spin 1 ferromagnetic atoms in a two-dimensional homogeneous condensate. This idealized model accurately represents a gas in a trap, narrow in the z direction, and much wider than a vortex core in the x and y directions. The spinor $\psi(x, y)$ describing the condensate has the Hamiltonian $\mathcal{H} = \iint dxdy(\frac{\hbar^2}{2m}|\nabla\psi|^2 + \mathcal{V})$; the potential energy is

$$\mathcal{V} = \tfrac{1}{2}\alpha{:}(\psi^\dagger\psi)^2{:} + \tfrac{1}{2}\beta{:}(\psi^\dagger \boldsymbol{S}\psi)^2{:} + q\psi^\dagger S_z^2\psi - \mu|\psi|^2. \tag{1}$$

Here α and β describe the mean and spin-dependent interaction parameters, and q and μ are the quadratic Zeeman energy and chemical potential, respectively.

Spin vortices occur in ferromagnetic condensates (defined by $\beta < 0$ [5]), and they are not stable without the quadratic Zeeman term. This can be produced by applying a magnetic field along z. The ground states of the condensate have the form

$$\psi = e^{i\theta_C + i\theta_S S_z}\begin{pmatrix}\sqrt{n_1}\\ \sqrt{n_0}\\ \sqrt{n_{-1}}\end{pmatrix} = \begin{pmatrix} e^{i(\theta_C+\theta_S)}\sqrt{n_1}\\ e^{i\theta_C}\sqrt{n_0}\\ e^{i(\theta_C-\theta_S)}\sqrt{n_{-1}}\end{pmatrix}. \tag{2}$$

The values n_1, n_0, n_{-1} correspond to the optimal proportions of the three spin states, and are fixed (the total density is $n = n_1 + n_0 + n_{-1}$). If the magnetization along the z-axis, $M_z = n_1 - n_{-1}$, vanishes, then the spin is in the xy-plane, as is preferred by the quadratic Zeeman effect. The overall phase of the wave function (which is not observable) is θ_C, and $-\theta_S$ is the azimuthal angle of the spin. (To see this, calculate $\langle S_x\rangle$ and $\langle S_y\rangle$ for this state.) The latter angle can be measured by scattering polarized light off the condensate.

There are two types of vortices in a spinor condensate in a magnetic field: a charge vortex, described by $\theta_C = \pm\phi$, $\theta_S = 0$ and a spin vortex described by $\theta_C = 0$, $\theta_S = \pm\phi$ (and observed in a rubidium-87 condensate [9]). Here, ϕ is the azimuthal angle centered on the vortex core. Figure 1 illustrates these vortices:

$$\psi_C = \begin{pmatrix} e^{i\phi}\sqrt{n_1}\\ e^{i\phi}\sqrt{n_0}\\ e^{i\phi}\sqrt{n_{-1}}\end{pmatrix}, \qquad \psi_S = \begin{pmatrix} e^{i\phi}\sqrt{n_1}\\ \sqrt{n_0}\\ e^{-i\phi}\sqrt{n_{-1}}\end{pmatrix}. \tag{3}$$

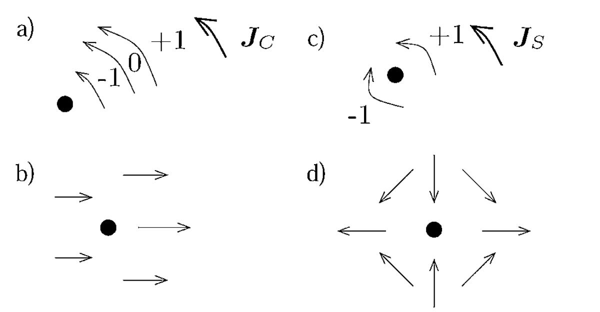

FIG. 1. Charge and spin vortices of strength 2π. (a),(c) The currents of the spin states around charge and spin vortices. (b), (d) The spin textures around charge and spin vortices.

The spin texture is uniform around a charge vortex (except near the core); the spin direction rotates by 360° clockwise or counterclockwise in a spin vortex.

A spinor condensate is similar to a mixture of several species of atoms. From this perspective, a spin vortex is a bound state of two opposite vortices in different components, and a charge vortex is a bound state of three vortices.

The velocity fields, $\frac{\hbar}{M}\nabla\theta_m$ [6], in the components of a spinor condensate are related:

$$\boldsymbol{u}_1 = \boldsymbol{u}_C + \boldsymbol{u}_S; \qquad \boldsymbol{u}_0 = \boldsymbol{u}_C; \qquad \boldsymbol{u}_{-1} = \boldsymbol{u}_C - \boldsymbol{u}_S; \quad (4)$$

the flow in the middle component is the mean of the flows in the other two components. Vortices can be classified by the amounts θ_C and θ_S wind by, Q_C and Q_S respectively. Near a charge vortex, $Q_S = 0$ and $\boldsymbol{u}_S = 0$, since θ_S is constant. Hence all the atoms move in the same direction [see Fig. 1(a)], and there is a net transport of mass, called the charge current $\boldsymbol{J}_C$. On the other hand, near a spin vortex there is no net flux of atoms $\boldsymbol{J}_C$ if the magnetization vanishes, since the atoms of spin ±1 move at equal and opposite speeds. Such a flow does have a nonzero current of spin, $\boldsymbol{J}_S$. (If $M_z \neq 0$, charge and spin vortices each produce both kinds of currents: $J_C = n\boldsymbol{u}_C + M_z\boldsymbol{u}_S$ and $J_S = q_z\boldsymbol{u}_S + M_z\boldsymbol{u}_C$ where $q_z = \psi^\dagger S_z^2\psi$.)

Now the force on a vortex moving at speed $\boldsymbol{v}$ and tossed about by a flow of charge and spin is given by $\dot{\boldsymbol{p}} = -\hbar\hat{z} \times [\sum n_m Q_m(\boldsymbol{u}_m - \boldsymbol{v})]$. The term for a given value of m describes the lift force on the vortex in that component: according to Bernoulli's principle higher velocities correspond to lower pressures, so the vortex moves to the side where its velocity field is parallel to the relative velocity of the fluid. For a spin vortex of charge Q_S, the previous expression for the Bernoulli forces can be summed and written in the form

$$\dot{\boldsymbol{p}} = \boldsymbol{F}_S - Q_S\boldsymbol{v} \times \hbar M_z\hat{z}. \quad (5)$$

Here the force $\boldsymbol{F}_S$ is proportional to the background spin current and the second term is the Magnus force, or lift, responding to the vortex's own motion. For a charge vortex of charge Q_C, the force (if $M_z = 0$) is $\dot{\boldsymbol{p}} = \boldsymbol{F}_C - Q_C\boldsymbol{v} \times \hbar n\hat{z}$, where $\boldsymbol{F}_C$ is produced by charge current.

Idiosyncrasies of vortex motion.—Let us compare the motion of charge and spin vortices. A charge vortex in stationary fluid cannot drift along a straight line, because the lift force would push it sideways. Furthermore, the equations of motion take the form of first order differential equations for charge vortices when the inertial term $\dot{\boldsymbol{p}}$ is assumed to be small:

$$\frac{d\mathbf{r}}{dt} = \frac{1}{n\hbar Q_C}\hat{z} \times \boldsymbol{F}_C; \quad (6)$$

motion is perpendicular to applied forces. The vortex velocity required by Eq. (6) turns out to equal the background flow speed, in accordance with Descartes's conception of planetary motion. The motion of charge vortices is determined once their initial positions are given.

In contrast, a spin vortex behaves in a Newtonian way, as long as the condensate has zero magnetization. First, in the absence of spin current, Eq. (5) implies "Newton's first law of spin vortices": a spin vortex in a charge current can move at any constant speed. There is no lift to push the vortex off course because the component vortices rotate in opposite directions [see Fig. 1(c)]. Second, if there is a spin current the lift forces on the component vortices from the counterpropagating flows add to produce a nonzero $\boldsymbol{F}_S$. The solution to $\dot{\boldsymbol{p}} = 0$, $|\boldsymbol{v}| = \frac{|F_S|}{\hbar Q_S M_z}$, does not make sense if $M_z = 0$. Therefore the inertial term cannot be neglected and "Newton's second law of spin vortices" results: a spin-vortex in a spin current must accelerate, at a rate proportional to $\boldsymbol{F}_S$. One can introduce the vortex mass m_v by assuming that

$$\boldsymbol{p} = m_v\dot{\boldsymbol{r}}, \quad (7)$$

at least at low speeds. Hence, the equation of motion prescribes the acceleration:

$$m_v\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{F}_S. \quad (8)$$

These Newton's laws describe spin vortices in condensates of any spin, as long as $M_z = 0$ [10]. The expression for the spin force is analogous to electrostatics:

$$\boldsymbol{F}_{S12} = \frac{\hbar^2 q_z}{M}\frac{Q_{S1}Q_{S2}\hat{\boldsymbol{r}}_{12}}{r_{12}}. \quad (9)$$

Phase space and equations of motion.—The equation of motion of the condensate

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2\nabla^2}{2M}\psi + \frac{\partial\mathcal{V}}{\partial\psi^\dagger} \quad (10)$$

has the form of Hamilton's equations, with ψ and $\psi^\dagger$ conjugate variables. [See Eq. (1) for the definition of $\mathcal{V}$.] The laws of motion for a set of charge and spin vortices follow from this; their dynamics can be described by differential equations which are also first order in time. One can concentrate on a few variables u^α describing the state of just the vortices, and not other details of the wave functions. The equations for these variables also have the form of Hamilton equations, coming in conjugate pairs.

The necessary variables for N charge vortices are their $2N$ spatial coordinates, and the first order differential equation for the evolution of these variables is Eq. (6). This equation fits into the framework of classical mechanics, even though there are no momentum coordinates. The y coordinate of each vortex (times nh) acts as the momentum conjugate to x [11]. Consequently, charge-vortex gases can have strange thermodynamics [12,13].

The phase space for N spin vortices must be $4N$ dimensional, since the equations of motion are second order. What can the extra $2N$ variables be? For ordinary objects, they are momenta. But momenta exist only in abstract space: a photograph of a ball in the air does not reveal its destination. A spin vortex's speed is independent of its position as well. But because Eq. (10) is a first order differential equation with respect to time, a single photograph of a vortex must have a tell-tale feature which can be used to deduce the vortex's velocity.

To guess what this feature is, note that the lift forces on the component vortices of a moving spin vortex pull them in opposite directions, but the components are restrained from drifting apart completely (see below). Thus, the stretching of the spin vortex increases with its speed (see Fig. 2). The momentum $\boldsymbol{p}$ can be discerned from a snapshot if it is proportional to the stretching:

$$\boldsymbol{p} = -hn_1\hat{z} \times \boldsymbol{D}, \tag{11}$$

where $\boldsymbol{D}$ points between the component vortices. This quantity is conjugate to the position. To understand this intuitively [14], suppose the component vortices are point-vortices with conjugate x and y coordinates, like charge vortices. If the one with circulation $\pm\frac{h}{m}$ is at $(x_\pm, y_\pm)$, then the Poisson brackets are $\{x_\pm, y_\pm\} = \pm\frac{1}{n_1 h}$. Short calculations using this show that the center of mass x is conjugate to $p_x = n_1 h D_y = n_1 h(y_+ - y_-)$ (i.e., $\{x, p_x\} = 1$) rather than to y. For realistic, spread-out vortices, $\boldsymbol{D}$ has to be $\frac{m}{n_1 h}\iint d^2\boldsymbol{u}\boldsymbol{r}(\mathrm{curl}\boldsymbol{J_C})_z$, the dipole moment of the vorticity (see [15]).

To derive the conjugacy relations in general, one has to evaluate the matrix of Poisson brackets between all pairs of variables. The entries in the inverse of this matrix (the "Lagrange brackets") can be expressed by integrals, $[u, v] = -2\hbar\Im\iint\frac{\partial\psi^\dagger(\boldsymbol{r};u,v)}{\partial u}\frac{\partial\psi(\boldsymbol{r};u,v)}{\partial v}d^2\boldsymbol{r}$ where $\psi(\boldsymbol{r}; u, v)$ is parameterized by the coordinates (u, v) of interest. Stokes's theorem implies that $[x, y] = -nm\kappa_C$, where κ_C is the circulation of $\boldsymbol{J}_C$. Hence, for example, x and y are conjugate for a charge vortex, but not for a spin vortex at zero magnetization since $\kappa_C = \frac{M_z}{nm}Q_S = 0$.

Since $\boldsymbol{r}$ and $\boldsymbol{p}$ are conjugate, the Hamilton equation for $\dot{\boldsymbol{p}}$ gives the force law [16]. The Hamilton equation $\dot{\boldsymbol{r}} = \frac{\partial\mathcal{H}}{\partial\boldsymbol{p}}$ relates the canonical momentum (the vortex distortion) to actual motion. Only the core energies $E_c(D)$ of the vortices depend on the distortion, so $\boldsymbol{v} = \frac{\partial E_c(\boldsymbol{D})}{\partial\boldsymbol{p}}$.

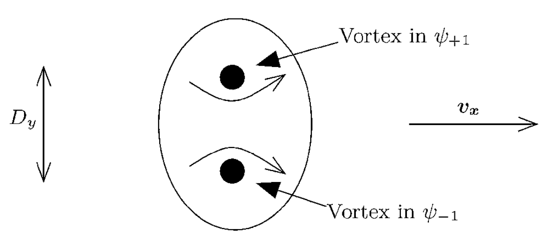

FIG. 2. Seeing the momentum of a spin vortex. The phases of $\psi_{\pm1}$ increase in the directions of the arrows. The component vortices are pulled apart by the lift force to a distance $D_y \propto p_x$. The phase locking produced by the spin-dependent interaction limits the stretching, since the phase λ is nonzero over the region indicated by the ellipse.

Confinement and the vortex mass.—Spinor condensates can have higher-order symmetries that do not occur for ordinary mixtures (e.g., of two different atoms) [17]. But the higher symmetry is not the only special thing about spinor condensates. Unlike in a mixture, the atomic states can turn into one another in a spinor condensate, as long as the angular momentum does not change: $1 + -1 \leftrightarrow 0 + 0$. This coherent-spin flipping is included among the interaction terms in the potential energy [Eq. (1)], as can be seen by writing them in the form $\mathcal{V}_{\mathrm{int}} = \frac{1}{2}\sum_{m_1,m_2}\alpha_{m_1m_2}{:}|\psi_{m_1}|^2|\psi_{m_2}|^2{:} + 2\beta\Re(\psi_1^\dagger\psi_{-1}^\dagger\psi_0^2)$, where $\alpha_{m_1m_2}$, functions of α and β, describe the interactions between pairs of spin species.

Coherent spin flipping locks the phases of the spinor components together [18,19] and this keeps the two parts of a spin vortex bound together. A nonzero phase $\lambda = (\theta_1 + \theta_{-1} - 2\theta_0)$ costs energy

$$\mathcal{E}_{\mathrm{flip}} = 2\beta n_0\sqrt{n_1 n_{-1}}\cos\lambda, \tag{12}$$

where the spin-dependent interaction parameter β is negative. Since the energy cannot depend on a phase unless particle numbers are not conserved, it makes sense that this comes from the spin-flipping reactions.

The energy of a stretched spin vortex is

$$E \sim |\beta|n^2D^2. \tag{13}$$

By reinterpreting this as kinetic energy $\frac{p^2}{2m_v}$ with the help of Eq. (11) we now obtain the vortex mass

$$m_v \sim \frac{\hbar^2}{\beta} \sim M\frac{w}{\Delta a}. \tag{14}$$

The second expression is obtained by relating β to the width of the condensate w in the z direction and the scattering-length difference Δa [19]. In this case, m_v is of the same order as the mass of the atoms in the vortex core, nl_m^2M, where l_m is the size of the core of a spin vortex, about $\sqrt{\frac{w}{n\Delta a}}$. For rubidium in a 1 μm wide trap with $\Delta a \sim 1$ Å, $m_v \sim 10^{-21}$ kg. I hope to calculate the numerical coefficient in Eq. (14) and its dependence on $\frac{q}{n_0|\beta|}$ in a future article [20].

Measuring the mass.—Now consider the consequences of spin vortices' inertial motion. Spin vortices of opposite signs can orbit around one another whereas opposite charge vortices always push each other along parallel lines [1]. As for planets, the orbits have different shapes depending on the initial momenta of the vortices. When the two vortices move on a circle the period is proportional to the circle's radius. Balancing the attraction Eq. (9) against the centrifugal force $|\dot{\boldsymbol{p}}| = p\omega$ gives

$$vp(v) = \frac{2\pi n_1\hbar^2}{M}, \tag{15}$$

determining the speed $v = v_{\mathrm{circ}}$ of the vortices. For any orbit radius v_{circ} is the same. Measuring v_{circ} helps to

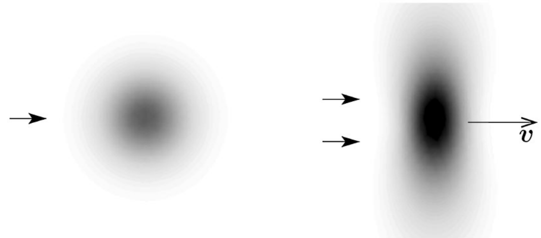

FIG. 3. The energy density around vortex cores moving at $v = 0, .45\sqrt{\frac{\mu}{M}}$, with rotationally symmetric interaction parameters (see text). These profiles are obtained from numerical steady-state solutions to the two-dimensional Gross-Pitaevskii equation, Eq. (10). Blacker regions have greater energy densities. The small arrows indicate the roots of ψ_1 and ψ_{-1}.

estimate the vortex mass, by assuming $m_v = \frac{p(v_{\mathrm{circ}})}{v_{\mathrm{circ}}}$, but v_{circ} is too large for the dispersion to remain linear.

A more accurate way to measure the mass of spin vortices is to observe their motion when the magnetization is not zero, but is small. Then the lift force due to the vortex's motion looks like the Lorentz force from a small magnetic field, $\boldsymbol{B}_{\mathrm{eff}} = -M_z \hbar \hat{z}$. A single vortex will therefore follow cyclotron orbits with the period

$$\tau = \frac{m_v}{\hbar M_z}. \tag{16}$$

If the magnetization is 5% and the other condensate parameters are as above, the period comes out as 0.3 sec.

Limiting velocity.—A final idea for an experiment is to study the motion of a rapidly moving vortex. Such a measurement allows hypothetical inhabitants of the superfluid to measure the velocity of their "ether." Spin vortices can move inertially only up to a certain velocity relative to the condensate.

Imagine pushing a spin vortex, starting from rest. After a certain amount of acceleration, the vortex may start to oscillate, so that all the additional work goes into producing spin waves. Alternatively, the vortex may remain stable. Then the work goes into stretching the components apart until the vortex becomes needle-shaped. The velocity is bounded in this case as well. E is proportional to the vortex length $D \propto p$ and a linear dispersion implies a finite velocity.

Spin-wave dissipation seems to be the fate of a vortex in a condensate with rotationally symmetric interactions [of form $\frac{1}{2}\alpha(\psi^\dagger\psi)^2 + \frac{1}{2}\beta(\psi^\dagger \mathbf{S}\psi)^2$]. This conclusion is based mainly on numerical solutions of the Gross-Pitaevskii equations for steadily moving vortices with $\beta = -0.3\alpha$, $q = 0.5\mu$ (see Fig. 3). The computer did not find solutions past $v_c = 0.65\sqrt{\frac{\mu}{M}}$, which is close to the spin-wave speed.

Experimental conditions can be adjusted so that a stretched vortex does not radiate energy. The traps for the three S_z states should be displaced into parallel planes in order to make the three spin states more independent. (The clouds still have to overlap to allow spin flipping.) The component vortices can then separate to great distances, approaching the speed $\frac{4n_0}{\pi}\sqrt{\frac{|\beta|}{nM}}$ [20].

To summarize, each charge vortex has only spatial degrees of freedom because the lift force overcomes the inertia. In contrast, spin vortices in an unmagnetized condensate behave like classical particles because they are made up of oppositely rotating vortices whose total Magnus force cancels. The internal stretching between these components give rise to the mass. Spin vortices at low speeds have 4 degrees of freedom: the center of mass coordinates and the momenta which are proportional to the distortion of the vortex core. The vortices betray their composite origin when accelerated sufficiently.

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