Nonequilibrium Dynamics of Weakly and Strongly Paired Superconductors

V. Gurarie

Department of Physics, University of Colorado, Boulder, Colorado 80309, USA

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We study small oscillations of the order parameter in weakly and strongly paired superconductors driven slightly out of equilibrium, in the collisionless approximation. While it was known for quite some time that the amplitude of the oscillations in a weakly paired superconductor decays as $t^{-1/2}$, we show that in a superconductor sufficiently strongly paired so that its fermions form bound states usually referred to as molecules, these oscillations decay as $t^{-3/2}$. The transition between these two regimes happens when the chemical potential of the superconductor vanishes; thus, the behavior of the oscillations can be used to distinguish weakly and strongly paired superconductors. Finally, we interpret the result in the strongly paired superconductor as the probability of the molecular decay as a function of time.

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The study of quantum quenches, or the evolution of a quantum system when its parameters suddenly change, acquired prominence due to recent studies in Refs. [1,2]. Yet a particular case of it, the study of small oscillations of the order parameter in a perturbed superconductor, started more than 30 years ago in Ref. [3]. This paper showed that if a small perturbation is applied to a superconductor leading to a deviation of its gap from its equilibrium value, then this perturbation will evolve as $\cos(2\Delta_0 t)/\sqrt{t}$, where Δ_0 is the equilibrium value of the gap. Thirty years later, the authors of Ref. [4] showed that large oscillations of the order parameter in a superconductor do not decay but in fact continue forever (as long as the collisionless approximation remains valid). Subsequently, with the help of the exact integrability of this problem [5], it was shown that there exists a critical strength of the perturbation such that if a perturbation beyond that strength is applied, it excites oscillations which continue forever while perturbations below that strength still decay, Refs. [6,7].

At the same time, advances in atomic physics allow now to create superconductors out of ultracold fermionic atoms whose interactions can be externally controlled. If the interactions are weak, these artificial superconductors behave just like the usual superconductors of condensed matter physics (termed BCS superconductors). But the interactions can also be adjusted to be strong, in which case these superconductors become more akin to a Bose condensate (BEC) of diatomic bosonic molecules. As the interactions are tuned, the superconductor is said to undergo a BCS-BEC crossover (which is accompanied by the chemical potential changing its sign). The possibility of this crossover was discussed in a number of papers throughout the last four decades (see Refs. [8–10]), and it finally was observed a few years ago (see Refs. [11,12]).

The unprecedented control over the interactions that the ultracold gases provide allows to change them quickly, thus easily creating an initial out-of-equilibrium perturbation by a sudden change of their strength. A natural question which arises in this regard concerns the fate of the oscillations, decaying and persistent, as the superconductor undergoes the BCS-BEC crossover. References [13,14] explored the large nondecaying oscillations of a tunable superconductor using the ansatz of Ref. [4]. However, this method was found to break down as the superconductor is tuned to the BEC regime, and it was not possible to tell whether this was an artifact of the technique or an indication that an oscillating solution is indeed not possible in a strongly paired superconductor (see, however, Ref. [15]).

To remedy this situation, in this Letter we consider small oscillations of the order parameter in a tunable superconductor. We show that as long as the oscillations remain small, they always decay regardless of the strength of interactions in the superconductor. However, while in the BCS superconductor, the amplitude of these oscillations decay as $1/\sqrt{t}$; in the BEC regime they decay as $1/t^{3/2}$. The transition between the two regimes happens exactly where the chemical potential μ is equal to zero.

The theory we develop is based on the mean field (collisionless) approximation. Thus, one can question its validity near the unitary point [16] which lies between the BCS and BEC regimes of the superconductors (at positive chemical potential) and is known to go beyond the applicability of the mean field theory [17] (see also Ref. [18] for recent studies of the dynamics near this point). Therefore, we repeat the calculation in the two-channel model [19] describing BCS-BEC crossover with narrow Feshbach resonance [20] where mean-field theory is applicable throughout and recovers the same result. This shows that an example of the BCS-BEC crossover exists whose order parameter oscillations unambiguously obey the scenario discussed here.

In the "deep BEC" side of the crossover, the superconductor becomes the Bose-Einstein condensate of the diatomic bosonic molecules. In this regime, the result $1/t^{3/2}$ has a very simple interpretation. Indeed, as we will see in this Letter, the order parameter of such superconductor is proportional to the probability of finding two fermions of opposite spins at the same position in space. As the superconductor is perturbed, the wave function of a pair of fermions no longer coincides with their bound state, but rather consists of the linear combination of a bound and excited states. The part of the wave function in the excited states moves off to infinity so that the probability amplitude to find two particles in the same spot decreases as $1/t^{3/2}$ reaching a limiting finite value at large t which in turn follows from the behavior of a three dimensional propagator of a free particle. In fact, in this deep BEC regime, this is true not only with a small but also with a relatively large initial perturbation (the precise criteria are developed below), in which case this picture makes it clear that the order parameter decays to a value smaller than the equilibrium value, just like in the BCS regime with certain perturbations [7]. However, to fully analyze the case of large perturbations to the superconductor and to see whether nondecaying solutions are possible in the BEC regime, one needs to take advantage of the integrability of the equations of motion, something which was done successfully for the weakly paired superconductor and whose strongly paired superconductor applications are left for future work.

In what follows, we describe the derivation of these results. Consider spin-1/2 fermions interacting via a short range attractive s-wave interaction of strength λ. Within the mean field approximation, these fermions can be studied on a strictly classical level. We accomplish this by introducing the Cooper pair number $n_p = \langle \hat{a}^\dagger_{\uparrow p} a_{\uparrow p} + \hat{a}^\dagger_{\downarrow p} a_{\downarrow p}\rangle/2$. The evolution of this number obeys the following classical Hamiltonian, a classical version of the Anderson-Richardson (or reduced BCS) Hamiltonian describing these kind of fermions,

$$H = 2\epsilon_p n_p - \frac{\lambda}{V}\sum_{p,q}\sqrt{n_p(1-n_p)n_q(1-n_q)}\cos(\phi_p - \phi_q), \tag{1}$$

where $\epsilon_p = p^2/(2m)$ is the free fermion dispersion, ϕ_p is the phase variable canonically conjugate to n_p, and V is the space volume. The equations of motion of this Hamiltonian have the following solution, corresponding to a stationary superconductor

$$\phi_p^0 = -2\mu t, \qquad n_p^0 = \frac{1}{2}\left(1 - \frac{\xi_p}{E_p}\right), \tag{2}$$

where as always μ is the chemical potential, t is time, $\xi_p = \epsilon_p - \mu$, $E_p = \sqrt{\xi_p^2 + \Delta_0^2}$, and finally Δ_0 is the equilibrium gap of the superconductor which can be found from the gap equation $1 = \frac{\lambda}{2V}\sum_p \frac{1}{E_p}$. Now consider the initial conditions for the motion described by Eq. (1) consisting of a small perturbation to the stationary solution Eq. (2). In fact, it is most physical to take as initial conditions Eq. (2) for a slightly different value of $\Delta = \Delta_0 + \delta\Delta_0$ as well as the slightly different value of μ (corresponding to a superconductor whose interactions were slightly perturbed in the initial moment of time). Then it is straightforward to see, with the help of the particle conservation condition $\sum_p \delta n_p = 0$, that the initial conditions read

$$n_p = n_p^0 + \delta n_p^0, \qquad \delta n_p^0 = \left(\xi_p - \frac{\tilde{f}_0}{f_0}\right)\frac{\Delta_0 \delta\Delta_0}{2E_p^3}, \tag{3}$$

where f_0 and $\tilde{f}_0$ are defined below in Eq. (9). We then expand the Hamiltonian Eq. (1) about the stationary solution to obtain the quadratic Hamiltonian for the deviations δn_p, $\delta\phi_p$. Subsequently we construct the solution to the equations of motion of this Hamiltonian by using the method of Green's functions. The expanded Hamiltonian takes the form

$$\delta H = \frac{1}{2}\sum_{p,q}[\delta\phi_p \Phi_{pq}\delta\phi_q + \delta n_p K_{pq}\delta n_q], \tag{4}$$

where

$$\Phi_{pq} = \frac{\Delta_0^2}{E_p}\delta_{pq} - \frac{\lambda}{2V}\frac{\Delta^2}{E_p E_q}, \qquad K_{pq} = \frac{4E_p^3}{\Delta_0^2}\delta_{pq} - \frac{2\lambda\xi_p\xi_q}{V\Delta_0^2}. \tag{5}$$

This represents a collection of harmonic oscillators, labeled by the index p. Now it is possible to construct a retarded Green's function corresponding to these oscillators, with the end result

$$\delta n_p(t) = i\sum_q \int_{-\infty+i0}^{\infty+i0}\frac{\Omega d\Omega}{2\pi}G_{pq}(\Omega)e^{-2i\Omega t}\delta n_q^0, \tag{6}$$

where

$$G_{pq}(\Omega) = \frac{\delta_{pq}}{\Omega^2 - E_p^2} + \frac{\lambda}{2V} \times \frac{f[\Omega^2\xi_p\xi_q + E_q^2(\Omega^2 - \Delta_0^2)] - \tilde{f}(\xi_p E_q^2 + \xi_q\Omega^2)}{\Omega^2 E_p(\Omega^2 - E_p^2)(\Omega^2 - E_q^2)[\tilde{f}^2 + f^2(\Delta_0^2 - \Omega^2)]}. \tag{7}$$

Here, we introduced the functions

$$f = \frac{\lambda}{2V}\sum_p \frac{1}{E_p(\Omega^2 - E_p^2)}, \qquad \tilde{f} = \frac{\lambda}{2V}\sum_p \frac{\xi_p}{E_p(\Omega^2 - E_p^2)}. \tag{8}$$

In what follows, we will also need these same functions evaluated at $\Omega = 0$, or

$$f_0 = -\frac{\lambda}{2V}\sum_p \frac{1}{E_p^3}, \qquad \tilde{f}_0 = -\frac{\lambda}{2V}\sum_p \frac{\xi_p}{E_p^3}. \tag{9}$$

The derivation of Eq. (7) is technical and not particularly instructive, so it will be published elsewhere.

Armed by the explicit expression for the fluctuations $\delta n_p(t)$, we can now discuss how to calculate the time dependent gap function. It is easiest to study the square of its absolute value,

$$|\Delta(t)|^2 = \frac{\lambda^2}{V^2}\sum_{p,q}\sqrt{n_p(1-n_p)n_q(1-n_q)}\cos(\phi_p - \phi_q). \tag{10}$$

In turn, this quantity can be decomposed into the sum of Δ_0^2, the square of the unperturbed gap, and the perturbation $\delta|\Delta|^2$, given by

$$\delta|\Delta(t)|^2 = \frac{2\lambda}{V}\sum_p \xi_p \delta n_p(t). \tag{11}$$

We combine this with the initial conditions Eq. (3) as well as with Eqs. (6) and (7) to find that

$$\delta|\Delta(t)|^2 = \frac{i\Delta_0\delta\Delta_0}{\pi}\int_{-\infty+i0}^{\infty+i0}\Omega d\Omega e^{-2i\Omega t} \times \frac{\tilde{f}^2 + f^2(\Delta_0^2 - \Omega^2) - ff_0\Delta_0^2 - f\tilde{f}_0^2/f_0}{\Omega^2[\tilde{f}^2 + f^2(\Delta_0^2 - \Omega^2)]}. \tag{12}$$

This equation represents the main result of this Letter. While reducing to the main result of Ref. [3] at weak pairing, it represents the generalization of their result to a superconductor of an arbitrary pairing strength. The integration here goes over the straight line slightly above the real axis, as shown on Fig. 1 with a dashed line.

We expect the functions f and $\tilde{f}$ to have singularities on the real axis where $|\Omega|$ hits the minimum of E_p and cuts at larger $|\Omega|$. It may seem that there is also a singularity at $\Omega = 0$, but it is actually fictitious as the numerator of Eq. (12) vanishes at $\Omega \to 0$. At $t < 0$, the contour can be closed in the upper half plane resulting in zero for the integral as expected. At positive time $t > 0$, we can deform the contour as shown on Fig. 1 by a solid line with arrows. Then, at large positive times, the main contribution to the integral comes from the vicinity of the singular points, or the turning points of the new contour.

To evaluate f and $\tilde{f}$, we replace summation over p in Eq. (8) by the integration over $Vd^3p/(2\pi)^3$. In the BCS regime, it is standard to pass to the integration over the variable $\xi = \epsilon_p - \mu$ so that $d^3p/(2\pi)^3 \approx \nu d\xi$, where $\nu = \sqrt{m^3\mu}/(\pi^2\sqrt{2})$ is the density of states, and then extend the integration over ξ all the way to the entire real axis. Then $\tilde{f} = 0$ as its integral is antisymmetric in ξ. At the same time, f is given by

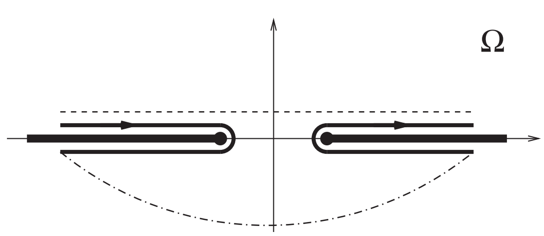

FIG. 1. The contour of integration in Eq. (12) in the complex plane of Ω. Dashed line shows the initial contour, while solid black line, together with a dashed-dotted semicircle, is the deformed contour. The dashed-dotted part of the new contour can be deformed away to infinity and does not contribute.

$$f = i\frac{\nu\lambda}{\Omega\sqrt{\Delta_0^2 - \Omega^2}}\ln\left[\sqrt{1 - \frac{\Omega^2}{\Delta_0^2}} + i\frac{\Omega}{\Delta_0}\right]. \tag{13}$$

It is then easy to see that the integrand in Eq. (12) has a singularity as Ω approaches Δ_0 which goes as $1/\sqrt{\Omega - \Delta_0}$ in agreement with the discussion above (Δ_0 coincides with the minimum of E_p). Introducing the variable $s = \Omega - \Delta_0$ and taking into account that at $t \gg 1/\Delta_0$ only small s contributes to the integral, we find after adding both left and right contours (which are complex conjugate of each other)

$$\delta|\Delta(t)|^2 \sim \mathrm{Re}\, e^{-2i\Delta_0 t}\int_0^\infty \frac{dse^{-ist}}{\sqrt{s}} \sim \frac{\cos(2\Delta_0 t)}{\sqrt{t}}. \tag{14}$$

Alternatively, in the BEC regime where $\mu < 0$, the integration measure in Eq. (8) must be kept as $Vd^3p/(2\pi)^3$ and both functions f and $\tilde{f}$ are not zero. The minimum of E_p is equal to $E_{\min} = \sqrt{\mu^2 + \Delta_0^2}$. The functions f and $\tilde{f}$ are now finite when $|\Omega|$ reaches $E_{\min}$, but they are not regular at that point. Generally, they behave as $f \sim \text{const} + i\sqrt{\Omega - E_{\min}}$. The origin of such a drastically different behavior lies in the fact that the minimum of E_p is reached when $p = 0$. Then, the potential divergence in the integral at $\Omega = E_{\min}$ is removed by the p^2 coming from the measure of integration p^2dp. All this can be checked explicitly in the deep BEC regime where $|\mu| \gg \Delta_0$ and the analytic expressions behaving precisely in this way can be obtained by direct integration.

Now if we substitute this into Eq. (12), we find that the constant piece is single valued as Ω goes from the upper to lower branch of the contour on Fig. 1, and therefore its contribution from each of the contours cancels. So, it is the square root part which contributes. This gives at large times $t \gg 1/E_{\min}$

$$\delta|\Delta|^2 \sim \mathrm{Re}\, e^{-2itE_{\min}}\int dse^{-ist}\sqrt{s} \sim \frac{\cos(2tE_{\min})}{t^{3/2}}. \tag{15}$$

As we see, the transition from the behavior Eq. (14) and (15) occurs exactly at $\mu = 0$ as this is the point where the minimum of the spectrum E_p shifts to $p = 0$.

One can worry if this derivation really works for a superconductor in the vicinity of the unitary point where the mean-field theory breaks down. To have a controllable theory, we repeated this calculation for the two-channel model, based on the equations of motion introduced in Ref. [13]. That model has an additional coupling g such that if $g^2m^2/n^{1/3} \gg 1$ (here n is the density), then the two-channel model is equivalent to Eq. (1) considered earlier in this Letter. In the opposite limit, the two-channel model still undergoes a crossover but with the mean field theory applicable throughout. The answer for the two-channel model can be worked out using the same methods as the ones described here. It looks almost identical to Eq. (12), except in all occurrences of $\tilde{f}$ and $\tilde{f}_0$, one needs to substitute $\tilde{f} \to \tilde{f} - 2\lambda/g^2$, $\tilde{f}_0 \to \tilde{f}_0 - 2\lambda/g^2$. This replacement does not change any of the arguments presented here for the one-channel model; thus, all the conclusions remain valid.

Finally, we observe that in the deep BEC regime, all $n_p^0 \ll 1$. Then the Hamiltonian Eq. (1) can be significantly simplified, reducing in this limit to the Schrödinger equation of one pair of fermions in a delta-function potential, as discussed at length in Ref. [10]. Replacing $1 - n_p \approx 1$, and introducing the pair wave function $\psi_p = \sqrt{n_p} e^{i\phi_p}$, we find $H = 2\epsilon_p |\psi_p|^2 - \frac{\lambda}{V} |\sum_p \psi_p|^2$ leading to

$$i\partial_t \psi = 2\epsilon_p \psi_p - \frac{\lambda}{V} \sum_q \psi_q. \tag{16}$$

The "gap function" in this language is $\Delta = \lambda \sum_p \psi_p / V$ which has an obvious meaning of the probability of finding two opposite-spin fermions at the same point in space. We can solve the linear equation (16) with arbitrary initial conditions (as long as they satisfy $|\psi_p^0| \ll 1$) directly. The solution for the gap function reads

$$\begin{aligned} \Delta(t) &= i \frac{\lambda}{V} \sum_p \int_{-\infty + i0}^{\infty + i0} \frac{d\Omega}{2\pi} e^{-i\Omega t} \frac{1}{1 + \Pi} \frac{\psi_p^0}{\Omega - 2\epsilon_p}, \\ \Pi &= \frac{\lambda}{V} \sum_p \frac{1}{\Omega - 2\epsilon_p} = \text{const} + \frac{\lambda m^{3/2} \sqrt{-\Omega}}{4\pi} \end{aligned} \tag{17}$$

with ψ_p^0 being the initial value of ψ_p. To compute the integral over Ω, we deform the contour of integration in a way similar to that shown on Fig. 1. Unlike Eq. (12), the integrand here has a simple pole at a negative value of $\Omega = \Omega_b$ where $\Pi(\Omega_b) = -1$, but similarly to Eq. (12) in the BEC regime, it has cut at $\Omega > 0$ where it behaves as const $+ i\sqrt{\Omega}$. The pole corresponds to the bound state and results in a contribution to $\Delta(t)$ corresponding to the bound molecule (that is, the equilibrium solution). The cut produces the decaying behavior going precisely as $1/t$; thus, overall $|\Delta(t)|^2 \approx |\Delta_f|^2 + A\cos(\Omega_b t)/t^{3/2}$ where A is some constant [for comparison with Eq. (15), observe that $\Omega_b = 2\mu$ in this regime]. However, now we see that this decay is correct for arbitrary initial conditions, as long as the applicability conditions $|\psi_p^0| \ll 1$ are satisfied. As noted earlier, the value Δ_f which the gap function decays to can be proven to be smaller than the equilibrium value of the gap. Indeed, initially we can expand $\psi_p^0 = \alpha_0 \psi_p^{(0)} + \sum_n \alpha_n \psi_p^{(n)}$ where $\psi_p^{(0)}$ is the bound state wave function, $\psi_p^{(n)}$ are the wave functions of the excited states corresponding to the unbounded motion, and obviously $|\alpha_0| < 1$. At large times, the excited states move off to spacial infinity and only the bound state contributes to $\sum_p \psi_p$, leading to $\Delta_f = \alpha_0 \Delta_e$, where $\Delta_e = \lambda \sum_p \psi_p^{(0)}/V$ is the equilibrium gap function, thus concluding the proof. This should be contrasted with the method of small perturbations used earlier in this Letter, which despite the statements in Ref. [3] to the contrary cannot be used to deduce whether the final gap is smaller or larger than its equilibrium value.

In conclusion, we showed that the amplitude of the small oscillations of the order parameter in a superconductor decays as $1/t^{1/2}$ in the BCS ($\mu > 0$) and as $1/t^{3/2}$ in the BEC ($\mu < 0$) regimes. Applicability of this result to the Fermi gas close to the unitary point remains an open question. In cold atomic systems, these oscillations can in principle be detected with the rf-absorbtion techniques such as the ones used in Ref. [21]. It would also be very interesting to see if, with modern STM-based techniques, these oscillations can be detected in real superconductors and whether this theory can be generalized to superconductors with the *d*-wave symmetry of the gap.

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