Turbulent Transfer of Energy by Radiating Pulses

Benno Rumpf
Physics Institute, Chemnitz University of Technology, 09107 Chemnitz, Germany

Alan C. Newell and Vladimir E. Zakharov
Department of Mathematics, University of Arizona, Tucson, Arizona 85721, USA
(Received 3 June 2009; published 14 August 2009)

We propose a new mechanism for turbulent transport in systems which support radiating nonlinear solitary wave packets or pulses. The direct energy cascade is provided by adiabatically evolving pulses, whose widths and carrier wavelengths decrease. The inverse cascade is due to the excitation of radiation. The spectrum is steeper than the Kolmogorov-Zakharov spectrum of wave turbulence.

DOI: 10.1103/PhysRevLett.103.074502 PACS numbers: 47.27.E−, 47.27.De, 52.35.−g, 92.10.−c

Wave turbulence is a highly successful theory for turbulent nonequilibrium processes in plasmas, fluids, and nonlinear optics. It provides an analytic connection of the deterministic nonlinear dynamics to statistical properties of a turbulent energy flow [1]. Its description of weakly interacting waves with almost random phases culminates in a kinetic equation for the wave action density. The Kolmogorov-Zakharov (KZ) solutions to this kinetic equation describe the transfer of energy (wave action) from long (short) scales to short (long) scales. The results of Cai and co-workers [2], designed to check wave turbulence theory, threw down the gauntlet for advocates of its universal applicability. They studied a one-dimensional model

$$i\dot{\psi}(x,t) = \mathcal{L}\psi(x,t) + \lambda\psi(x,t)|\psi(x,t)|^2, \qquad (1)$$

where $\psi(x,t)$ is a complex wave amplitude and the linear operator $\mathcal{L}$ is defined by $\mathcal{L}\exp(ikx) = \omega_k \exp(ikx)$ with a square-root dispersion $\omega_k = \sqrt{|k|}$ designed to mimic deep water waves. The Majda, McLaughlin, and Tabak (MMT) equation (1) derives from the Hamiltonian $E = E_2 + E_4$, $E_2 = \int \omega_k |a_k|^2 dk$ and $E_4 = \lambda \int |\psi|^4/2dx$, with Fourier modes $a_k = \int \psi(x,t)\exp(-ikx)dx/\sqrt{2\pi}$. Conservation of wave action $N = \int |\psi|^2 dx$ and momentum $P = i\int(\psi\psi_x^* - \psi_x\psi^*)dx$ are related to the phase and translational symmetries of (1). A statistically stationary nonequilibrium state is achieved when external damping is applied at very long scales and at short scales, and driving is applied at long scales. This causes two conserved density cascades, a direct cascade of energy and an inverse cascade of wave action. In wave turbulence both cascades are driven by the same four wave resonances producing long and short waves. The KZ spectrum of the wave action density for the direct cascade is $\langle |a_k|^2 \rangle \sim k^{-1}$ in wave turbulence, which is independent of the sign λ of the nonlinearity. Repeated trials of careful experiments [2–5] showed that the KZ spectrum is recovered for the MMT equation with $\lambda = -1$. For $\lambda = 1$, one finds a steeper spectrum of roughly $k^{-1.25}$ (Fig. 1). This leads to the following intriguing questions: What new mechanisms are responsible for energy and wave action transfer? What causes the failure of wave turbulence theory?

In this Letter, we propose a new mechanism of turbulent transfer that is radically different from that of wave turbulence. In repeated simulations of (1), the most striking feature is a spatiotemporal pattern of left- and right-moving localized structures (Fig. 2). Their speeds decrease during their lifetime so that the traces are curved (Fig. 3). We suggest that these evolving coherent wave packets (pulses) cause the cascades of wave action and energy. Their spectral width is initially of the same order as their central wave number so that they have few oscillations and a large central peak (Fig. 4). They resemble giant (freak) waves encountered in the ocean [6] and in optical fibers [7]. These pulses are initiated by a Benjamin-Feir instability, and they arise only in the MMT equation with $\lambda = 1$. They emit Cherenkov-like radiation as the solitary "head" resonantly drives long oscillatory "tails" [4,8,9]. The pulse heads change adiabatically, because they lose wave action

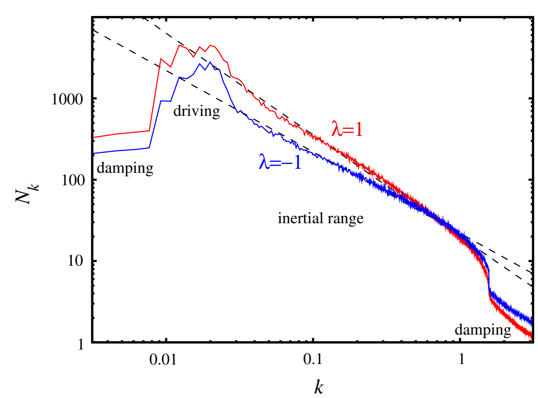

FIG. 1 (color online). Time-averaged power spectra $N_k = \langle |a_k|^2 \rangle$ for the MMT equation (1). The equation with $\lambda = -1$ yields a Kolmogorov-Zakharov spectrum $\langle |a_k|^2 \rangle \sim k^{-1}$, and $\lambda = 1$ yields a steeper spectrum $\langle |a_k|^2 \rangle \sim k^{-1.25}$. The system contains $L = 4096$ nodes, and the wave number space is $-\pi < k \leq \pi$. Damping is applied to the modes $|k| \leq 10\pi/L$ and $|k| \geq \pi/2$, and the modes $20\pi/L < |k| \leq 30\pi/L$ are driven externally.

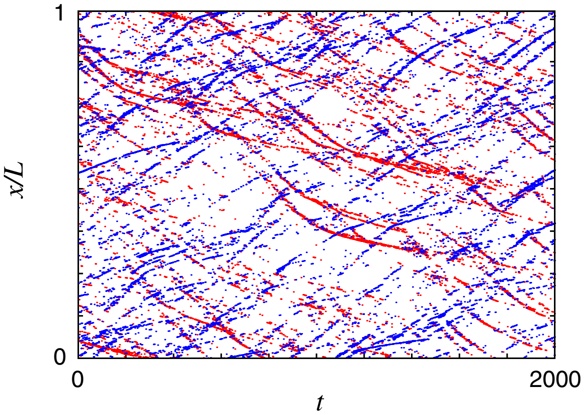

FIG. 2 (color online). Pattern of left- and right-moving pulses in time and space with periodic boundary conditions for the MMT equation with $\lambda = 1$. The system is damped and driven as in Fig. 1. Pulses appear spontaneously and cross each other without significant loss of power. The speed of the pulses decays in time.

and energy to their resonant radiation tails. We will show that the direct energy cascade is carried by gradually deforming pulse heads. The inverse cascade is carried by radiation to the tails. Our simulations make it clear that pulses interact very little with each other and change adiabatically. Therefore the field ψ is dominated by an ensemble of noninteracting and evolving pulses whose statistical properties can be computed as a weighted time average over the history of a single pulse. We can predict the spectrum by computing the adiabatic change of the wave packet which agrees very closely with what we observe. This dynamics supercedes four wave resonances of weak turbulence.

Figure 3 shows the formation of a pulse from an initial long wave. Such pulses arise from a Benjamin-Feir–type instability of a monochromatic wave solution $\psi = \psi_0 \exp(ik_m x)$ of (1). Setting $\psi = (\psi_0 + \delta a)\exp(ik_m x)$ with $\delta a = \delta a_+ \exp(iqx) + \delta a_- \exp(-iqx)$ the frequency σ of $\delta a_\pm$ is $\sigma = \pm\sqrt{M^2} \pm \sqrt{D(2|\psi_0|^2 + D)}$ with $D = (\sqrt{|k_m + q|} + \sqrt{|k_m - q|} - 2\sqrt{|k_m|})/2$, $M = (\sqrt{|k_m + q|} - \sqrt{|k_m - q|})/2$. For $\lambda = 1$, σ can have an imaginary part corresponding to an instability. The most unstable sideband q is at k_m.

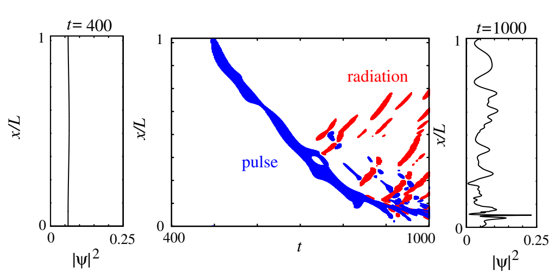

FIG. 3 (color online). Time evolution for the initial condition of $\psi(x, t = 0) = 0.25\exp(i2\pi x/L)$ plus weak noise for the MMT equation ($\lambda = 1$) with no damping and driving. The figure shows traces of regions with high momentum density. A pulse appears by the Benjamin-Feir instability of the initial wave. Its speed decays in time as it emits counterpropagating radiation.

This nonlinear pulse corresponds to an extremum $E^{(f)}$ of the energy for given values of the momentum $P^{(f)}$ and the wave action $N^{(f)}$. It can be computed with Lagrange multipliers from $d(E - \Omega N - vP) = 0$. A special case is the quasisoliton solution $\psi^{(\mathrm{sol})}(x, t) = q\sqrt{-\omega_m''}\,\mathrm{sech}[q(x - \omega_m' t)]\exp[i(k_m x - \omega_m t + \omega_m'' q^2 t/2)]$, which is valid for almost monochromatic wave packets with $q \ll k_m$ and $\omega_m = \omega(k_m)$ [4]. Here $\omega_m' = d\omega/dk|_{k_m}$ is the group velocity. In contrast to the Benjamin-Feir instability of the nonlinear Schrödinger equation, the wave packet emerging from the instability of (1) has q and k_m of the same order. The pulse contains essentially only one loop, and it can be written approximately as $\psi^{(f)}(x, t) = q\sqrt{\omega_m} k_m^{-1} f(\theta) \times \exp(i\alpha)\exp(i\Omega t)$ where f, θ, and α are real functions with $\partial\theta/\partial x = q$, $\partial\theta/\partial t = -qv$, $\partial\alpha/\partial x = k_m$, $\partial\alpha/\partial t = -k_m v$. The phase frequency in the frame that moves with the speed $v = \partial E^{(f)}(N^{(f)}, P^{(f)})/\partial P^{(f)}$ is $\Omega = \partial E^{(f)}(N^{(f)}, P^{(f)})/\partial N^{(f)}$. In Fourier space, (1) is

$$i\dot{a}_k - \omega_k a_k = T_k, \tag{2}$$

where the nonlinearity T_k for a pulse solution is given by

$$T_k = \int_{-\infty}^{\infty} \psi^{(f)}|\psi^{(f)}|^2 \exp(-ikx)dx/\sqrt{2\pi} \sim q^2 k_m^{-9/4} F_k \exp(-i\Lambda_k t), \tag{3}$$

and F_k is the Fourier transform of $f^3\exp(i\alpha)$. The Doppler-shifted frequency in (3) is $\Lambda_k = \Omega + kv$. Modes a_k in (2) are driven by the time-dependent force T_k. A mode with a k value outside the pulse will respond strongly if it is in resonance $\omega_k = \Lambda_k$ with the oscillatory frequency of T_k. The pulse excites the linear wave at $k = k_{\mathrm{res}} < 0$ for $k_m > 0$ (Fig. 4). This causes the radiation of wave action, energy, and negative momentum from the pulse to a long wave. We assume that a pulse emits an amount of wave action $dN^{(\mathrm{rad})} > 0$. Wave action, momentum, and energy of the pulse and the radiation are balanced by the conserva-

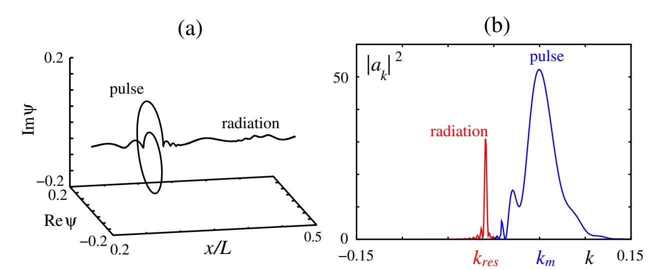

FIG. 4 (color online). (a) Pulse and low-amplitude long wave radiation in space. (b) In Fourier space, the pulse has the maximum power at k_m. The radiation is a narrow peak at $k_{\mathrm{res}} \approx -(\sqrt{2} - 1)^2 k_m$.

tion equations $dN^{(f)} + dN^{(\mathrm{rad})} = 0$, $dE^{(f)} + dE^{(\mathrm{rad})} = 0$, $dP^{(f)} + dP^{(\mathrm{rad})} = 0$. Multiplying the resonance condition with the pulse's decrement of wave action yields $(\omega_{\mathrm{res}} - \Omega - v k_{\mathrm{res}}) dN^{(f)} = 0$, which is equivalent to the extremum condition of the pulse $d(E - \Omega N - vP) = 0$. In other words, the radiating pulse remains an extremum of the energy with respect to wave action and momentum. We can obtain a general equation for the relation between $P^{(f)}$ and $N^{(f)}$ as follows. Since $dE^{(\mathrm{rad})} = \sqrt{|k_{\mathrm{res}}|}dN^{(\mathrm{rad})}$, $dP^{(\mathrm{rad})} = k_{\mathrm{res}} dN^{(\mathrm{rad})}$, we find $(dE^{(\mathrm{rad})}/dN^{(\mathrm{rad})})^2 = -(dP^{(\mathrm{rad})}/dN^{(\mathrm{rad})})$. From the conservation laws, this translates to

$$(dE^{(f)}(N^{(f)}, P^{(f)}(N^{(f)}))/dN^{(f)})^2 = -dP^{(f)}/dN^{(f)}. \quad (4)$$

The exact expression $E^{(f)}(N^{(f)}, P^{(f)})$ is unknown, but the numerical evidence (Fig. 5) shows that E_2 is very much larger than E_4, and that the ratio $\sqrt{P^{(f)}N^{(f)}}/E^{(f)}$ is close to 1. Solving Eq. (4) with the approximated energy function $E^{(f)} \approx \sqrt{P^{(f)}N^{(f)}}$ yields $P^{(f)} \sim N^{(f)\sqrt{8}-3}$ and $E^{(f)} \sim N^{(f)\sqrt{2}-1}$, which is verified by simulations of Fig. 5(b).

This means that the pulse frequency $\Lambda_m = \Omega + k_m v$ may be approximated by $\omega_m = \sqrt{k_m}$, and its wave action $N^{(f)}$ by $bqk_m^{-3/2}$ with $b = \int_{-\infty}^{\infty} f^2 dx$. Its momentum $P^{(f)}$ is approximately $k_m N^{(f)}$ and its energy $E^{(f)}$ is approximately $\sqrt{k_m}N^{(f)}$. The decrements of the share of these quantities are connected to changes of $q > 0$ and $k_m > 0$ by

$$\begin{aligned} dN^{(f)} &= bk_m^{-3/2}(dq - \tfrac{3}{2}qk_m^{-1}dk_m), \\ dE^{(f)} &= bk_m^{-1}(dq - qk_m^{-1}dk_m), \\ dP^{(f)} &= bk_m^{-1/2}(dq - \tfrac{1}{2}qk_m^{-1}dk_m). \end{aligned} \quad (5)$$

The energy of the radiation at k_{res} is $dE^{(\mathrm{rad})} = \gamma\sqrt{k_m}dN^{(\mathrm{rad})}$, and its momentum is $dP^{(\mathrm{rad})} = -\gamma^2 k_m dN^{(\mathrm{rad})}$ with $\gamma = \sqrt{|k_{\mathrm{res}}/k_m|}$. The pulse loses wave action and energy, but gains momentum in this evolution. Expressing $dN^{(\mathrm{rad})}$ with $dN^{(f)}$ of (5) yields

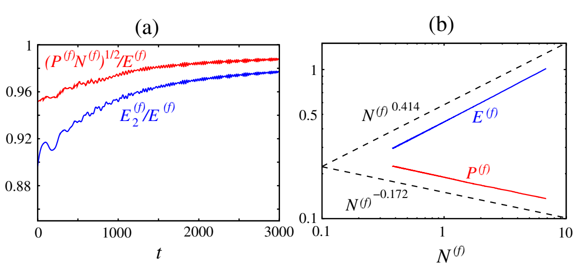

FIG. 5 (color online). (a) The ratios $(P^{(f)}N^{(f)})^{1/2}/E^{(f)}$ and $E_2^{(f)}/E^{(f)}$ of an evolving pulse as functions of time. (b) Energy $E^{(f)} \sim N^{(f)\sqrt{2}-1}$ and the momentum $P^{(f)} \sim N^{(f)\sqrt{8}-3}$ of an evolving pulse as functions of the wave action $N^{(f)}$. The wave action and the energy decrease in time, while the momentum increases.

$$\begin{aligned} dE^{(\mathrm{rad})} &= -\gamma bk_m^{-1}(dq - \tfrac{3}{2}qk_m^{-1}dk_m), \\ dP^{(\mathrm{rad})} &= \gamma^2 bk_m^{-1/2}(dq - \tfrac{3}{2}qk_m^{-1}dk_m). \end{aligned} \quad (6)$$

Energy conservation with (5) and with (6) yields $[(3\gamma - 2)/(2\gamma - 2)]dk_m/k_m = dq/q$, while momentum conservation gives $[(1 + 3\gamma^2)/(2 + 2\gamma^2)]dk_m/k_m = dq/q$. These expressions match for $\gamma = \sqrt{2} - 1$, or $k_{\mathrm{res}} = -(\sqrt{2} - 1)^2 k_m$. As a result, the width and the carrier wave number of a radiating pulse are related by

$$q(t) = q(t_0)[k_m(t)/k_m(t_0)]^\eta, \quad (7)$$

with $\eta = (3\gamma - 2)/(2\gamma - 2) \approx 0.646$. Both $k_m(t)$ and $q(t)$ increase as time evolves, and the ratio q/k_m decays. The speed of a pulse decays in time (Figs. 2 and 3). The expression (7) is equivalent to the solution $P^{(f)} \sim N^{(f)\sqrt{8}-3}$ of (4).

The response of $a_k^{(\mathrm{rad})}$ at a fixed $k < 0$ to the driving force T_k can be computed from (2). A pulse at k_m drives this mode if k is close to $k_{\mathrm{res}} = -\gamma^2 k_m$. The frequency Λ_k has a slow time dependence due to the evolution of the pulse. When Λ_k is close to ω_k, it can be approximated by a linear chirp $\Lambda_k(t) \approx \omega_k + \dot{\Lambda}_k t$, where $\dot{\Lambda}_k \sim \dot{k}_m/\sqrt{k_m}$ is small and changes little near resonance. With $a_k^{(\mathrm{rad})} = b_k^{(\mathrm{rad})}\exp(-i\omega_k t)$, Eq. (2) is $i\dot{b}_k^{(\mathrm{rad})} = T_k \exp(-i\dot{\Lambda}_k t^2)$. Using $\int_{-\infty}^{\infty}\cos(wt^2)dt = \sqrt{\pi/(2w)}$, the amplitude of the mode at k will be $|a_k^{(\mathrm{rad})}|^2 \sim T_k^2/\dot{\Lambda}_k$ or

$$|a_k^{(\mathrm{rad})}|^2 \sim T_k^2\sqrt{k_m}/\dot{k}_m \quad (8)$$

after the driving frequency Λ_k has moved through resonance at $\Lambda_k = \omega_k$. The amplitude can also be computed from conservation of wave action. The pulse that moves from k_m to $k_m + dk_m$ radiates an amount of wave action $dN \sim bqk_m^{-5/2}dk_m$ into the interval $[-\gamma^2(k_m + dk_m), -\gamma^2 k_m]$. Therefore the modes in this interval have the amplitude

$$|a_k^{(\mathrm{rad})}|^2 \sim q(k_m)k_m^{-5/2}. \quad (9)$$

For the evolution of the carrier wave number, Eqs. (8) and (9) yield $\dot{k}_m \sim q^{-1}k_m^3 T_k^2$. For the quasisoliton, the feed T_k to the radiation mode is exponentially small in k_m/q. For a nonlinear pulse, k_m and q are of the same order, and the leading order $T_k \sim q^2 k_m^{-9/4}$ (3) yields $\dot{k}_m \sim q^3 k_m^{-3/2}$.

Averaging over the time history of a single pulse as its wave number increases yields

$$\begin{aligned} \langle|a_k^{(f)}|^2\rangle &= \int_{t_0}^{t_1}|a_k^{(f)}(k_m(t), q(k_m(t)))|^2 dt \\ &= \int_{k_m(t_0)}^{k_m(t_1)}|a_k^{(f)}(k_m, q(k_m))|^2/\dot{k}_m dk_m. \end{aligned} \quad (10)$$

$a_k^{(f)}(k_m, q)$ is the mode at k for a pulse that is centered at k_m. The approximation $|a_k^{(f)}|^2 \approx \delta(k - k_m)N^{(f)}$ assumes that

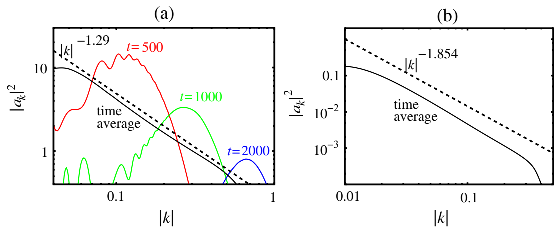

FIG. 6 (color online). (a) Time evolution of a radiating pulse. In wave number space the peak of intensity of the pulse at k_m moves towards higher k. The wave action of the pulse decays. The time average of the evolving pulse yields a spectrum $\langle |a_k^{(f)}|^2 \rangle \sim k^{-1.29}$. (b) The radiation emitted by the pulse has a spectrum $\langle |a_k^{(\mathrm{rad})}|^2 \rangle \sim k^{-1.854}$.

the whole wave action is localized at k_m, which yields $\langle |a_{k=k_m}^{(f)}|^2 \rangle = N^{(f)}/\dot{k}_m$. The peak of the pulse spends an amount of time $dt = dk_m/\dot{k}_m$ between k_m and $k_m + dk_m$. This yields the spectrum

$$\langle |a_k^{(f)}|^2 \rangle \sim qk^{-3/2}/(q^3 k^{-3/2}) \sim q^{-2} \sim k^{-1.29}. \quad (11)$$

Figure 6(a) shows snapshots of the spectrum of a pulse at three different times and the time-averaged spectrum. It shows that the central wave number k_m and the width of a pulse increase, while its power decreases. The time-averaged spectrum is in good agreement with the computed power law $k^{-1.29}$. This type of spectrum is obtained only for a certain window in wave number space. The relative width in wave number space decays, according to (7), as $q(t)/k_m(t) \sim k_m^{\eta-1}(t)$. As a consequence, the intensity of radiation decays and the pulse behaves more like a quasi-soliton, and moves very little in wave number space. The time-averaged spectrum decays less rapidly as a function of k for averages over a very long time.

The simulations also confirm the spectrum of the radiation (9). Figure 6(b) shows the spectrum of the radiated wave action at negative k. For this purpose, damping is applied to negative k so that the radiation is quickly dissipated. The time-averaged spectrum closely matches the prediction $\langle |a_k^{(\mathrm{rad})}|^2 \rangle \sim q|k|^{-5/2} \sim |k|^{-1.854}$ from (9). The radiation spectrum applies also for quasisolitons with small q/k_m.

We have found a turbulent process where radiating pulses dominate the spectral flow. An open question is, How might one have known, *a priori*, that the transfer mechanisms are dominated by evolving coherent objects in the MMT equation with $\lambda = 1$, and by four wave resonant interactions in the equation with $\lambda = -1$? From experience to date, coherent objects such as pulses form at all scales only in one-dimensional systems. In dimension two and higher, there are corrections to the KZ spectrum at very large and very small scales [10,11]. A forthcoming study of the breakdown of the wave turbulence picture in one-dimensional systems will investigate instabilities of the KZ spectrum under perturbations with correlations. But the challenge of finding *a priori* general criteria, given low-amplitude initial conditions, which distinguish the circumstances under which resonant waves or coherent objects dominate the long time dynamics is still open.

[1] V. E. Zakharov, V. Lvov, and G. Falkovich, *Kolmogorov Spectra of Turbulence* (Springer-Verlag, Berlin, 1992); K. Hasselmann, J. Fluid Mech. **12**, 481 (1962); V. E. Zakharov and N. N. Filonenko, Dokl. Akad. Nauk SSSR **170**, 1292 (1966) [Sov. Phys. Dokl. **11**, 881 (1967)]; D. J. Benney and P. G. Saffman, Proc. R. Soc. A **289**, 301 (1966); D. J. Benney and A. C. Newell, Stud. Appl. Math. **48**, 29 (1969).
[2] A. J. Majda, D. W. McLaughlin, and E. G. Tabak, J. Nonlinear Sci. **7**, 9 (1997); D. Cai, A. J. Majda, D. W. McLaughlin, and E. G. Tabak, Proc. Natl. Acad. Sci. U.S.A. **96**, 14 216 (1999); D. Cai and D. W. McLaughlin, J. Math. Phys. (N.Y.) **41**, 4125 (2000); D. Cai, A. J. Majda, D. W. McLaughlin, and E. G. Tabak, Physica (Amsterdam) **152–153D**, 551 (2001).
[3] V. E. Zakharov, P. Guyenne, A. N. Pushkarev, and F. Dias, Physica (Amsterdam) **152–153D**, 573 (2001).
[4] V. Zakharov, F. Dias, and A. Pushkarev, Phys. Rep. **398**, 1 (2004).
[5] B. Rumpf and L. Biven, Physica (Amsterdam) **204D**, 188 (2005).
[6] H. Tamura, T. Waseda, and Y. Miyazawa, Geophys. Res. Lett. **36**, L01607 (2009).
[7] D. R. Solli, C. Ropers, P. Koonath, and B. Jalali, Nature (London) **450**, 1054 (2007).
[8] V. I. Karpman and H. Schamel, Phys. Plasmas **4**, 120 (1997).
[9] N. Akhmediev and M. Karlsson, Phys. Rev. A **51**, 2602 (1995).
[10] L. Biven, S. V. Nazarenko, and A. C. Newell, Phys. Lett. A **280**, 28 (2001); A. C. Newell, S. V. Nazarenko, and L. Biven, Physica (Amsterdam) **152–153D**, 520 (2001); L. Biven, C. Connaughton, and A. C. Newell, Physica (Amsterdam) **184D**, 98 (2003).
[11] A. C. Newell and V. E. Zakharov, Phys. Lett. A **372**, 4230 (2008).