Landau-type Order Parameter Equation for Shear Banding in Granular Couette Flow

Priyanka Shukla and Meheboob Alam*

Engineering Mechanics Unit, Jawaharlal Nehru Center for Advanced Scientific Research, Jakkur PO, Bangalore 560064, India

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We show that a Landau-type "order-parameter" equation describes the onset of shear-band formation in granular plane Couette flow wherein the flow undergoes an ordering transition into alternate layers of dense and dilute regions of low and high shear rates, respectively, parallel to the flow direction. Even though the linear theory predicts the stability of the homogeneous shear solution in dilute flows, our analytical bifurcation theory suggests that there is a subcritical finite-amplitude instability that is likely to lead to shear-band formation in dilute flows, which is in agreement with previous numerical simulations.

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The understanding of the dynamical behavior of granular materials, a collection of macroscopic dissipative particles, remains a major challenge in condensed matter physics [1]. This is partly due to the unavailability of a unified theory of granular materials for different flow regimes. In the rapid flow regime [1], the pattern formation in various canonical flows (vibrated bed, inclined chute flow, Poiseuille flow, Couette flow, etc.) has received considerable attention during the last few years [2]. One prototype problem in this category is the granular plane Couette flow (PCF) which undergoes a series of bifurcations due to stationary and traveling wave instabilities [3]. When a dense granular material is sheared in shear-cell experiments [4], shearing remains confined to a narrow localized zone (i.e., a "shear band") near the walls and the rest of the material remains almost unsheared—this is the analog of the well-known "gradient-banding" (for which the shear rate is nonuniform along the gradient direction) in many complex fluids (e.g., in wormlike micelles, polymer solutions, colloidal suspensions, soft glasses, etc.; see [5,6]). Such shear banding [7,8], wherein the flow undergoes an ordering transition into alternate layers of dense and dilute particle regions of low and high shear rates, respectively, aligned along the gradient direction, has also been realized in the molecular dynamics simulations of granular PCF for a range of densities from dilute to dense flows in the rapid flow regime. Possible theoretical description of patterns in rapid granular flows in terms of order-parameter equations is the focus of this Letter.

In traditional fluid mechanics [9], the Ginzburg-Landau-type order-parameter theories have been greatly successful in uncovering a multitude of patterns in fluid flows. Moving onto the "dissipative" granular fluid, we ask: can we use order-parameter theories to describe the pattern formation scenario in granular systems? Can we derive the relevant order-parameter equations from a first principle theory? One specific goal of this Letter is to explore whether we could describe the shear-banding phenomenon in granular PCF via an order-parameter equation, and another goal is to ascertain its predictive power via a qualitative comparison with previous simulations of granular PCF. Our order-parameter theory encompasses a much broader perspective of describing shear banding in a variety of complex (non-Newtonian) fluids since it has been established [6,10] that the shear banding in many complex fluids originates from an intrinsic instability of the fluid as in the present case of sheared granular fluid. A unified order-parameter theory can be developed, provided appropriate constitutive relations (e.g., the Johnson-Segalman model [6]) for complex fluids are known.

We consider a two-dimensional flow of monodisperse granular materials (with particle diameter d and mass density $\rho = \rho_p \phi$, where ϕ is the volume fraction of particles and ρ_p being their material density) driven by two oppositely moving parallel walls along the streamwise x direction. The walls are moving in opposite directions with a velocity $U_w/2$ and h is the gap between two walls. Let us focus on the streamwise-independent $[\partial/\partial x(\cdot) = 0]$ equations, with u and v being the velocity components in the x and y directions, respectively, and T be the granular temperature. The dimensionless balance equations and the constitutive relations are written down in the supplementary information [11]. The inelastic nature of particle collisions [12] is reflected in the collisional dissipation term $[\mathcal{D} \propto (1 - e^2) \to 0$, as $e \to 1$, where e is the restitution coefficient]. We have verified that the predictions of our theory do not depend on the choice of a particular Navier-Stokes-order model [10] as long as the constitutive relations account for both kinetic and collisional contributions [12].

The undisturbed laminar flow, whose nonlinear stability we wish to examine, is the steady $[\partial/\partial t(.) = 0]$, fully developed $[\partial/\partial x(.) = 0]$ flow with no-slip and zero heat flux boundary conditions [11]. The resultant base flow, denoted by the underbar,

$$\underline{u}(y) = y, \qquad \underline{\phi}(y) = \text{const}, \qquad \underline{T}(y) = \text{const}, \qquad (1)$$

consists of uniform shear rate ($\gamma = d\underline{u}/dy = 1$) with constant density and constant granular temperature. Note that there are three control parameters: the dimensionless Couette gap $H = h/d$, the restitution coefficient e, and the average density $\int_{-1/2}^{1/2} \phi(y)dy \equiv \underline{\phi}$.

Nonlinear analysis.—All field variables (density, velocity, and granular temperature) and transport coefficients are decomposed into a base-state component and a perturbation. The nonlinear perturbation equations can be written as

$$\left(\mathbf{I}\frac{\partial}{\partial t} - \mathcal{L}\right)X = \mathbf{N_2}(X, X) + \mathbf{N_3}(X, X, X), \qquad (2)$$

where $X = [\phi', u', v', T]^{\mathrm{Tr}}$ represents the vector of perturbation state variables, the superscript Tr indicates the transpose, $\mathcal{L}(\frac{\partial}{\partial y}, \frac{\partial^2}{\partial y^2}; \underline{\phi}, \underline{u}, \underline{T}, \ldots)$ is the linear operator and $\mathbf{I}$ is the identity operator; $\mathbf{N}_2$ and $\mathbf{N}_3$ are the quadratic and cubic nonlinear terms [11], respectively.

Linear stability.—Neglecting nonlinearities, we obtain the well-studied linear stability problem [3]: $\frac{\partial X}{\partial t} = \mathcal{L}X$. Using the normal mode ansatz of the form $X(y, t) = X^{[1;1]}(y)e^{\omega t}$ yields the following differential eigenvalue problem

$$\mathbf{L}\, X^{[1;1]} = \omega \mathbf{I} X^{[1;1]}, \quad \mathbf{B}X^{[1;1]} = 0, \quad \text{at } y = \pm 1/2, \qquad (3)$$

where $\mathbf{L}(d/dy, d^2/dy^2; \underline{\phi}, \underline{u}, \underline{T}, \ldots)$ is the linear ordinary differential operator [3], $\mathbf{B}$ is the boundary operator, representing the boundary conditions [3,11], and the complex frequency ω $(=\omega_r + i\omega_i)$ is the eigenvalue. The growth rate of the disturbance is denoted by the real part of ω and the imaginary part of ω is its frequency. The flow is stable, unstable, or neutrally stable if ω_r is negative, positive, or zero, respectively.

Using the inner product of two complex functions, $\langle f, g\rangle = \int_{-1/2}^{1/2} \tilde{f}(y)g(y)dy$, with tilde denoting a complex conjugate, the corresponding adjoint eigenvalue problem is $\mathbf{L}^\dagger X^\dagger = \tilde{\omega}\mathbf{I}X^\dagger$, with boundary conditions being the same as in Eq. (3). For the present problem, we have verified that the adjoint operator $\mathbf{L}^\dagger$ is just the transpose of $\mathbf{L}(d/dy \to -d/dy, d^2/dy^2; \ldots)$.

Nonlinear stability and center manifold reduction.—The spectrum of the linear operator is composed into slow, critical, or center (i.e., the modes have growth rates close to zero) and fast or critical (i.e., the modes having large decay rates) modes. Since the domain of our plane Couette flow is bounded, $y \in (-1/2, 1/2)$, the spectrum is discrete, which has been verified by computing the eigenvalues of the linear operator $\mathbf{L}$ [3,11].

The center manifold theorem [13] states that the dynamics close to the critical situation is dominated by a finite number of critical modes, resulting in an effective low-dimensional dynamical system. Focusing on a single slow mode, therefore, the disturbance vector field $\mathbf{X}(y, t) = \Phi(y, t) + \Psi(y, t)$ is decomposed as a linear combination of the linear critical eigenfunction Φ and an infinite number of noncritical eigenfunctions Ψ.

In our analysis, the most unstable shear-banding mode [3] from the linear theory (3), $X^{[1;1]}(y)$, called the fundamental mode (linear eigenfunction), represents the critical mode:

$$\Phi(y, t) = A(t)X^{[1;1]}(y) + \tilde{A}(t)\tilde{X}^{[1;1]}. \qquad (4)$$

In the weakly nonlinear theory, the spatial variation of the critical mode is taken to be the same as that of the linear theory, but its temporal variation is nonexponential (unlike in linear theory) having a finite complex amplitude $A(t)$ whose dynamical equation is of interest here.

To proceed further, we follow two steps [9,13]: (i) expand $X(y, t) = \sum_k X^{(k)} + \tilde{X}^{(k)}$ into a generalized Fourier series, and (ii) the Fourier coefficients $X^{(k)}$ are expanded into a Taylor series in terms of perturbation amplitude $A(t)$:

$$X = \sum_{k=0}^{\infty} X^{(k)} + \tilde{X}^{(k)} = \sum_{k=0}^{\infty} A^k|A|^{n-k}X^{[k;n]} + \text{c.c.}, \qquad (5)$$

with $n \geq 1$ and c.c. denotes a complex conjugate. Here the summation convention is such that $0 \leq k \leq n$, and the superscript convention [11] of $X^{[k;n]}$ is defined such that $X^{[k;n]} = 0$ if $(k + n)$ is odd. The explicit expression for the critical mode $\Phi(y, t)$ as in Eq. (4) can be easily extracted from the Taylor expansion of the first mode $X^{(1)}$ [as defined in Eq. (5); see [11]].

Inserting (5) into Eq. (2) and equating the like-order terms, we obtain [11]

$$\left(\frac{d}{dt} - \omega\right)A(t)X^{[1;1]} = \text{ nonlinear terms}, \qquad (6)$$

$$\left(\mathbf{I}\frac{\partial}{\partial t} - \mathcal{L}\right)\Psi(y, t) = \text{ nonlinear terms}. \qquad (7)$$

The former (6) is the nonlinear evolution equation for the critical mode Φ, and the latter Eqs. (7), representing all noncritical modes Ψ, are called enslaved equations. Note that we have used (3) to obtain the second term on the left-hand side of Eq. (6).

Landau equation.—Taking the inner product of Eq. (6) with adjoint linear eigenfunction $X^\dagger$ and separating the like-power terms in amplitude, we arrive at the Landau equation for the perturbation amplitude $A(t)$:

$$\frac{dA}{dt} = c^{(0)}A + c^{(2)}A|A|^2 + \ldots, \qquad (8)$$

where we have used a normalization condition, $\langle X^\dagger, X^{[1;1]}\rangle = 1$. It is straightforward to verify from (8) and (6) that $c^{(0)} \equiv \omega$ is the linear eigenvalue. Similarly, the expression for the first Landau coefficient, $c^{(2)}$, can be identified as

$$c^{(2)} = \langle X^\dagger, \mathcal{H}(X^{[1;1]}, X^{[2;2]}, X^{[0;2]})\rangle = \int_{-1/2}^{1/2} \tilde{X}^\dagger \mathcal{H}\, dy, \qquad (9)$$

where $\mathcal{H}(\cdot, \cdot, \cdot)$ represents a combination of nonlinear terms [11].

It is clear from Eq. (9) that, in addition to knowing the fundamental mode $X^{[1;1]}$ and its adjoint $X^\dagger$, we need to determine $X^{[2;2]}$ (the second harmonic, originates at $O(A^2)$

from the interaction of the fundamental mode with itself) and $X^{[0;2]}$ (the distortion to the mean flow [11], which is always real, originates at $O(A\tilde{A})$ from the interaction of the fundamental mode with its complex conjugate, and this provides the first modification to the mean or base flow). From the enslaved Eqs. (7) at quadratic order $O(A^2)$, we obtain the following equation for the second harmonic:

$$(2\omega \mathbf{I} - \mathbf{L})X^{[2;2]} = \mathbf{N}_2(X^{[1;1]}, X^{[1;1]}). \qquad (10)$$

This can be solved for $X^{[2;2]}$ since the right-hand side of Eq. (10) is a known function of the fundamental mode $X^{[1;1]}$. We have verified that the second harmonic and the distortion to the mean flow are equal, $X^{[0;2]} = X^{[2;2]}$, for the shear-banding instability.

Neglecting nonlinear terms in (8), we obtain the well-known linear stability result, $A(t) \sim \exp(\omega t)$, of exponential growth which is valid at linear order, $O(A)$, in amplitude. In the following discussion, we decompose $c^{(n)}$ into real and imaginary parts: $c^{(n)} = a^{(n)} + ib^{(n)}$, with $n = 0, 2, 4, \cdots$; for example, $a^{(0)}$ and $b^{(0)}$ represent the growth rate and the frequency of the disturbance, respectively. For the present problem of shear-banding instability, it has been verified [3] that $b^{(0)} = 0$; i.e., the unstable eigenvalue is always real which implies that the related bifurcation (see below), if any, must be of pitchfork-type. We have also verified that $b^{(2)} = 0$; i.e., the first Landau coefficient is always real for this unstable mode.

The equilibrium amplitude ($\frac{dA}{dt} = 0$) of disturbance can be obtained by truncating (8) at the cubic order:

$$A_e = \pm\sqrt{-a^{(0)}/a^{(2)}}, \qquad (11)$$

with the third solution $A_e \equiv 0$ representing the base-state of uniform shear and constant density and granular temperature. It is clear that the finite-amplitude equilibrium solutions (11) exist iff $a^{(0)}$ and $a^{(2)}$ are of opposite sign. The sign of the real part of $c^{(2)}$ decides the nature of bifurcation: a positive value for $a^{(2)}$ denotes a subcritical bifurcation and its negative value denotes a supercritical bifurcation.

Phase diagram and bifurcation.— The differential eigenvalue problem (3) and the differential equation (10) have been discretized using the Chebyshev spectral method [3]; see [11] for details on numerical method.

The phase diagram, separating the zones of stability and instability by the neutral contour ($a^{(0)} = 0$, red line), in the (ϕ, H) plane is shown in Fig. 1(a) for a restitution coefficient of $e = 0.95$; the flow is unstable ($a^{(0)} > 0$) inside the red neutral line, and stable ($a^{(0)} < 0$) outside. With decreasing value of e, the neutral contour shifts towards the left (i.e., at lower H) and the growth rate increases [3], and hence the size of the unstable region in the (ϕ, H) plane increases (and the flow becomes more unstable) with increasing dissipation. It may be noted that the linear stability equations (3) admit analytical solution [3]:

$$(\phi^{[1;1]}, T^{[1;1]})(y) = (\phi_1, T_1)\cos k_m(y \pm 1/2), \qquad (12)$$

where $k_m = m\pi$ is the "discrete" wave number along y, with $m = 1, 2, \ldots$ being the mode number that tells us the number of zero-crossing of the density or temperature eigenfunctions along $y \in (-1/2, 1/2)$. The neutral contour in Fig. 1(a) corresponds to mode $m = 2$ for which a typical density eigenfunction is displayed in the inset in Fig. 1(a). This suggests that inside the red neutral contour the unstable shear flow will give birth to new solutions having modulated density profiles along the gradient (y) direction. Interestingly, the uniform shear flow [Eq. (1)] is linearly stable for $\phi < \phi_c^l = \min_{\forall H}\phi(a^{(0)} = 0) \approx 0.154$ [below the lower branch of the red neutral contour in Fig. 1(a)]. In contrast to this prediction of the linear theory, however, we note that such density-segregated solutions have been found in the molecular dynamic simulations of granular shear flow; see the snapshot of particle positions in Fig. 1(b).

Let us now turn to analyze the results of our nonlinear theory. Figure 1(c) shows the variations of the first Landau

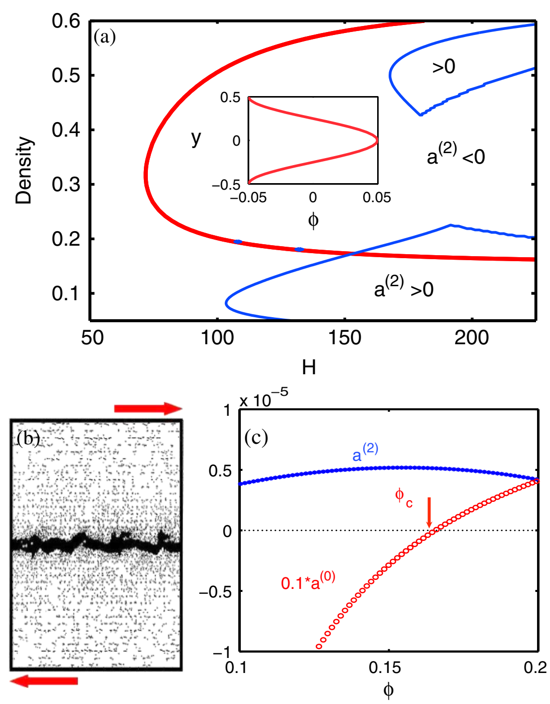

FIG. 1 (color online). (a) Phase diagram in the (H, ϕ) plane, showing the neutral stability contour (red line) and the contours of vanishing first Landau coefficient (blue lines) for a restitution coefficient of 0.95. The inset shows a typical density eigenfunction within the "linearly" unstable region. (b) Shear-band formation in simulations of dilute granular PCF: $\phi = 0.05$, $e = 0.6$ and $N = 20\,000$. (adapted from M.-L. Tan's thesis 1995). Lees-Edwards boundary condition has been used with the top boundary moving to the right and the bottom boundary to the left with the same speed. (c) Variations of $a^{(0)}$ and $a^{(2)}$ with density at $H = 200$ for $e = 0.95$. The critical density is defined as the one at which the linear growth rate is zero, $\phi_c = \phi(a^{(0)} = 0) \approx 0.157$ at $H = 200$.

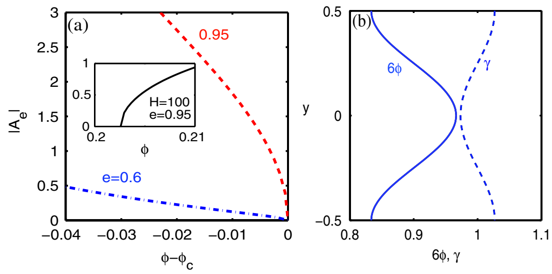

FIG. 2 (color online). (a) Bifurcation diagram in the amplitude-vs-density plane for $e = 0.95$ (red) and 0.6 (blue) at $H = 200$. Inset shows a bifurcation diagram at $H = 100$ and $e = 0.95$. (b) Finite-amplitude solutions for the density and the shear rate at $\underline{\phi} = 0.15$, $H = 200$ and $e = 0.6$. Note that the density has been multiplied by a factor of 6.

coefficient [$a^{(2)}$] and the linear growth rate [$a^{(0)}$] with mean density for $H = 200$ and $e = 0.95$. Since $a^{(2)} > 0$ and $a^{(0)} < 0$ for $\phi < \phi_c \equiv \phi(a^{(0)} = 0)$, the existence of finite-amplitude subcritical solutions is strongly suggested [viz. Eq. (11)] in the dilute limit.

The zero contour of the first Landau coefficient is superimposed in Fig. 1(a) as blue lines, and $a^{(2)} > 0$ inside the blue loops. As per Eq. (11), it is now clear that the finite-amplitude subcritical solutions are possible in the dilute limit, enclosed by the lower blue loop in Fig. 1(a). As in the case of the neutral contour [$a^{(0)} = 0$, the red contour in Fig. 1(a)], the blue contours for $a^{(2)} = 0$ shift towards the left with decreasing value of e (say, 0.6), and hence the size of the subcritical region (at dilute limit) in the (ϕ, H)-plane increases with increasing dissipation. This evidence of subcritical instability for dilute flows is in agreement with the simulations of Ref. [7].

With parameter values as in Fig. 1(c), the bifurcation diagrams in the $(A_e, \phi - \phi_c)$-plane are shown in Fig. 2(a) for two values of the restitution coefficient: $e = 0.95$ (red dash line) and 0.6 (blue dot-dash line). Each line in Fig. 2(a) provides the threshold-amplitude, $A_e(\phi, e)$, for nonlinear subcritical instability. This implies that if the perturbation amplitude A is larger than its threshold value [i.e., if $A > A_e(\phi, e)$], then the uniform shear flow will jump into a new state of nonuniform shear and nonuniform density across the gradient direction. Typical subcritical finite-amplitude solutions for the density (solid line), $\phi = \underline{\phi} + A_e \phi^{[1;1]}$, and the shear rate (dashed line), $\gamma = d/dy(\underline{u} + A_e u^{[1;1]})$, are displayed in Fig. 2(b), clearly showing density segregation and shear localization across the y direction; these shear-banded solutions have been calculated at the threshold amplitude $A = A_e$, with a mean density $\underline{\phi} = 0.15$, $H = 200$ and $e = 0.6$.

Figure 2(a) also suggests that the threshold amplitude for nonlinear instability decreases with increasing dissipation, implying that more dissipative particles are more prone to such subcritical shear-banding instability. The important point to note is that an appropriate magnitude of finite-amplitude perturbation, $A > A_e(\phi, e)$, must be imposed in simulations to achieve the shear-banded flow in the dilute limit.

At larger densities, the nature of bifurcation changes from subcritical to supercritical; see the inset of Fig. 2(a) for $H = 100$ and $e = 0.95$. The corresponding solutions for $\phi(y)$ and $\gamma(y)$ look similar to those in Fig. 2(b). Inside the upper loop of the blue contour in Fig. 1(a), we have $a^{(0)} > 0$ and $a^{(2)} > 0$, and hence finite-amplitude solutions do not exist [Eq. (11)] for some range of ϕ (inside upper blue loop) in the dense limit.

Conclusion.—Starting from the Navier-Stokes level continuum equations of inelastic dense-gas kinetic theory and using the center manifold reduction technique, we showed that a Landau-type order-parameter equation describes the shear-banding transition in granular PCF. Our results on the first Landau coefficient suggest that there is a subcritical finite-amplitude instability for dilute flows even though the dilute flow is stable according to the linear stability theory. The calculation of higher-order Landau coefficients (required to obtain the associated stable finite-amplitude solutions in the dilute limit) is left to a future work. Even though we focused on streamwise-independent flows in this Letter, our nonlinear order-parameter theory can be extended to analyze various nonlinear patterns in a host of granular flow problems as well as to describe shear banding in other complex fluids [5,6].

*Corresponding author.
meheboob@jncasr.ac.in

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