

Random Matrix Ensembles Associated with Lax Matrices

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A method to generate new classes of random matrix ensembles is proposed. Random matrices from these ensembles are Lax matrices of classically integrable systems with a certain distribution of momenta and coordinates. The existence of an integrable structure permits us to calculate the joint distribution of eigenvalues for these matrices analytically. Spectral statistics of these ensembles are quite unusual and in many cases give rigorously new examples of intermediate statistics.

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Introduction.—Statistical properties of surprisingly many different problems can be described by two main distributions: the Poisson statistics of independent random variables and the Random Matrix Theory (RMT) statistics. Classically integrable systems display spectral statistics which are close to Poisson distribution [1], but classically chaotic systems, in general, are well described by RMT [2]. The same classes of spectral statistics appear in the investigation of disordered systems, in particular, in the study of the 3-dimensional Anderson model (see, e.g., [3] and references therein). Though the universal character of the Poisson and RMT statistics is well established (but not always well understood), these two distributions do not exhaust all possible behaviors, even in the 3D Anderson model. A new phenomenon appears in this model when the disorder strength is set at a special value corresponding to the metal-insulator transition. In [4], it was established numerically that in this case, spectral statistics has features intermediate between Poisson and RMT behaviors. Similar hybrid statistics have been observed numerically [5] for certain dynamical systems which are neither integrable nor chaotic but belong to the class of pseudointegrable systems [6]. These and other examples demonstrate the possible existence of a not yet well-defined class of intermediate spectral statistics [3] with two characteristic features: level repulsion at small spacings as in RMT and exponential decrease of the nearest-neighbor distribution at large spacings as in the Poisson distribution.

The usual random matrix ensembles are chosen in such a way that their measure is invariant under conjugation $\mathcal{M} \rightarrow U\mathcal{M}U^{-1}$ over a group of either unitary, orthogonal, or symplectic matrices [7]. The invariance of eigenvalues of \mathcal{M} under these groups permits to find the exact joint distribution of all eigenvalues λ_j . In the simplest setting [7], the distribution reads

$$P(\boldsymbol{\lambda}) \sim \exp\left[-a \sum_k \lambda_k^2 + \beta \sum_{j < k} \ln|\lambda_j - \lambda_k|\right] \quad (1)$$

where $\beta = 1, 2, 4$ for, respectively, orthogonal, unitary, and symplectic ensembles.

For intermediate statistics, the situation is different. Physical problems giving rise to intermediate statistics have a natural basis in which they are defined, and in general do not possess explicit invariant measure, which makes the progress of their analytical treatment difficult.

In this Letter, we introduce new families of random matrix ensembles which are not invariant over geometrical transformations, but still allow us to obtain an exact joint distribution of eigenvalues analogous to (1). These ensembles give new nontrivial examples of intermediate statistics.

General construction.—To define our random matrix ensembles, we consider a classical one-dimensional N -body integrable model such that the equations of motion are equivalent to the matrix equation

$$\dot{L} = ML - LM \quad (2)$$

for a pair of Lax matrices L and M depending on momenta \mathbf{p} and coordinates \mathbf{q} [8].

It is the Lax matrix $L = L(\mathbf{p}, \mathbf{q})$ that we propose to consider as a random matrix depending on random variables \mathbf{p} and \mathbf{q} distributed according to a certain “natural” measure

$$dL = P(\mathbf{p}, \mathbf{q}) d\mathbf{p} d\mathbf{q} \quad (3)$$

which depends on the system. The only information we shall use from the integrability of the underlying classical system is the existence and explicit form of action-angle variables $I_\alpha(\mathbf{p}, \mathbf{q})$ and $\phi_\alpha(\mathbf{p}, \mathbf{q})$, and the identity

$$\prod_j dp_j dq_j = \prod_\alpha dI_\alpha d\phi_\alpha \quad (4)$$

due to the canonicity of the action-angle transformation. Direct proof of this key identity is difficult, and implicit methods were used to establish it [9,10]. Action variables turn out to be usually the eigenvalues λ_α of the Lax matrix or a simple function of them. The canonical change of

variables in (3) from momenta and coordinates to action-angle variables leads to a formal relation

$$dL = \mathcal{P}(\boldsymbol{\lambda}, \phi) d\boldsymbol{\lambda} d\phi. \quad (5)$$

The exact joint distribution of eigenvalues is then obtained by integration over angle variables, which can easily be performed in all cases considered:

$$P(\boldsymbol{\lambda}) = \int \mathcal{P}(\boldsymbol{\lambda}, \phi) d\phi. \quad (6)$$

This scheme is general and can be adapted to several different models. Because of space restrictions, we consider here only two representative ensembles, based on the rational Calogero-Moser (CM) [11] and the trigonometric Ruijsenaars-Schneider (RS) [12] models. Other examples and details of the calculations will be presented elsewhere [13].

Calogero-Moser ensemble.—The Hamiltonian of the rational CM model reads

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2} \sum_k p_k^2 + g^2 \sum_{i < j} \frac{1}{(q_j - q_i)^2} \quad (7)$$

with the following $N \times N$ Hermitian Lax matrix [14]

$$L_{jk} = p_j \delta_{jk} + ig \frac{1 - \delta_{jk}}{q_j - q_k}. \quad (8)$$

Let λ_α and $u_k(\alpha)$ be eigenvalues and right eigenvectors of L . In [9], it is proved that the matrix Q (called conjugate to L), defined by

$$Q_{\alpha\beta} = \sum_k u_k^*(\alpha) q_k u_k(\beta), \quad (9)$$

can be written as

$$Q_{\alpha\beta} = \phi_\alpha \delta_{\alpha\beta} - ig \frac{1 - \delta_{\alpha\beta}}{\lambda_\alpha - \lambda_\beta}, \quad (10)$$

where the new variables $\phi_\alpha = Q_{\alpha\alpha}$ are angle variables canonically conjugated to the action variables λ_α . Equation (10) is similar to (8) with the substitution $g \rightarrow -g$, $q_j \rightarrow$ eigenvalues λ_α and $p_j \rightarrow$ angle variables ϕ_α .

Consider now an ensemble of random matrices (8) defined by random variables p_i and q_j with measure

$$dL \sim \exp\left[-a \text{Tr} L^2 - b \sum_k q_k^2\right] d\mathbf{p} d\mathbf{q} \quad (11)$$

where a and b are positive constants. Using the described canonical change of variables and taking into account that $\sum_k q_k^2 = \text{Tr} Q^2$ with Q given by (9), the measure in (11) can be rewritten as

$$\begin{aligned} dL &\sim \exp[-a \text{Tr} L^2 - b \text{Tr} Q^2] d\boldsymbol{\lambda} d\phi \\ &= \exp\left[-a \sum_\alpha \lambda_\alpha^2 - b \sum_\alpha \phi_\alpha^2 - \sum_{\alpha \neq \beta} \frac{bg^2}{(\lambda_\alpha - \lambda_\beta)^2}\right] d\boldsymbol{\lambda} d\phi. \end{aligned} \quad (12)$$

Integration over ϕ gives a constant, and we are left with the

following exact joint distribution of eigenvalues of the Lax matrix L with measure (11)

$$P(\boldsymbol{\lambda}) \sim \exp\left[-a \sum_\alpha \lambda_\alpha^2 - bg^2 \sum_{\alpha \neq \beta} \frac{1}{(\lambda_\alpha - \lambda_\beta)^2}\right]. \quad (13)$$

According to this formula, eigenvalues of the above CM ensemble behave as a 1D gas of particles with inverse square interparticle potential. No long-range interaction proportional to $\ln|\lambda_i - \lambda_j|$ is present, in contrast with standard random matrix ensembles (1). We stress that our model is different from the Moshe-Neuberger-Shapiro model [15], which can be related to the Calogero-Sutherland model (see [16]).

The fast decrease of interparticle potential in (13) with the distance between particles permits to approximate (see, e.g., [5]) the nearest-neighbor distribution of eigenvalues of the Lax matrix (8) by the formula

$$P(s) \approx Ae^{-B^2/s^2 - Cs} \quad (14)$$

where B is a fitting constant, and constants A and C are determined from the normalization conditions. This expression is not exact but may be considered as an analog of the Wigner surmise in RMT [7].

The measure (11) for coordinates \mathbf{q} corresponds to N particles with repulsion confined in an interval of the order of $1/\sqrt{b}$. In order to simplify numerical investigation, it is therefore natural to use the "picket fence" approximation $q_k \sim k$ with integer k . We thus replace the matrix L by a simpler matrix

$$\tilde{L}_{jk} = p_k \delta_{jk} + ig \frac{1 - \delta_{jk}}{2(j - k)}. \quad (15)$$

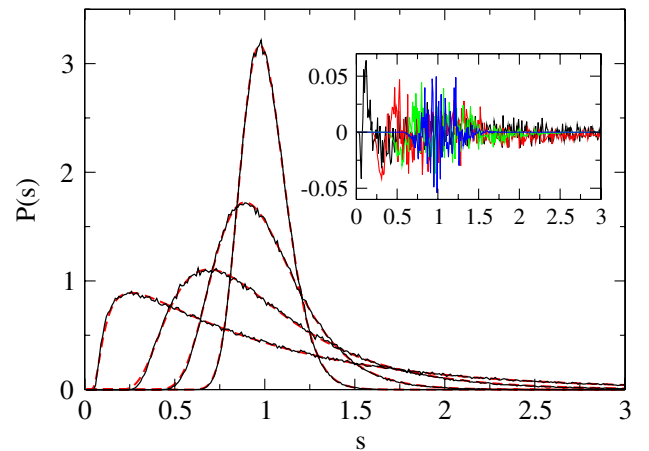


FIG. 1 (color online). Nearest-neighbor distribution for the CM random matrices (15) with $g = 0.1, 0.5, 1,$ and 2 (from left to right), averaged over 5000 realizations of matrices of size $N = 301$. Black solid lines are numerical results; red/gray dashed lines indicate the fit (14) with fitted values of B . For the considered values of g , $B = 0.096, 0.618, 1.46,$ and 3.11 , respectively. Inset: Difference between numerical result and fit for $g = 0.1$ (black), 0.5 (red/gray), 1 (green/light gray), and 2 (blue/dark gray).

The decrease as $|j - k|^{-1}$ of nondiagonal elements in \tilde{L} is a characteristic feature of intermediate systems [3]. In numerical calculations, we chose N variables p_k as i.i.d. random variables with uniform distribution between -1 and 1 . In Fig. 1, we show the nearest-neighbor spacing distribution for the matrix \tilde{L} for several values of the parameter g . The simple surmise (14) is practically indistinguishable from numerical results (see inset), which confirms the existence of unusual exponentially strong level repulsion in this model. We have checked that other distributions of the q_k lead to the same answer. The detailed comparison will be performed elsewhere [13].

Ruijsenaars-Schneider ensemble.—The second example we consider here is the trigonometric RS model [12]. It appears that the random matrix ensemble that was proposed in [17] as a quantization of a pseudointegrable interval-exchange map and investigated in [18,19] is a particular case of this model.

The RS model is determined by the Hamiltonian

$$H(\mathbf{p}, \mathbf{q}) = \sum_{j=1}^N \cos(p_j) V_j^{1/2}(\tau, \mathbf{q}) \quad (16)$$

where $V_j(\tau, \mathbf{q})$ depends on \mathbf{q} and a real parameter τ as

$$V_j(\tau, \mathbf{q}) = \prod_{k \neq j} \left(1 - \frac{\sin^2 \tau}{\sin^2[(q_j - q_k)/2]} \right). \quad (17)$$

The Lax matrix $L(\mathbf{p}, \mathbf{q})$ for this model is a unitary matrix given by

$$L_{jk}(\mathbf{p}, \mathbf{q}) = e^{i\tau(N-1)+ip_j} C_{jk}(\mathbf{q}) e^{i(q_k - q_j)/2} \quad (18)$$

[we choose a phase factor different from the one in [10] in order to get Eq. (29) when $q_k = 2\pi k/N$]. Here, $C(\mathbf{q})$ is the orthogonal matrix

$$C_{jk}(\mathbf{q}) = W_j^{1/2}(\tau, \mathbf{q}) \frac{\sin \tau}{\sin[(q_j - q_k)/2 + \tau]} W_k^{1/2}(-\tau, \mathbf{q}) \quad (19)$$

with $W_j(\tau, \mathbf{q}) W_j(-\tau, \mathbf{q}) = V_j(\tau, \mathbf{q})$ and

$$W_j(\tau, \mathbf{q}) = \prod_{k \neq j} \frac{\sin[(q_j - q_k)/2 + \tau]}{\sin[(q_j - q_k)/2]}. \quad (20)$$

Action-angle variables are obtained similarly as for the CM model. Here, one considers [10] the conjugate matrix Q defined by

$$Q_{\alpha\beta} = \sum_k u_k^*(\alpha) e^{iq_k} u_k(\beta) \quad (21)$$

where $u_k(\alpha)$ are eigenvectors of the Lax matrix (18) corresponding to eigenvalues $\lambda_\alpha = e^{i\theta_\alpha}$. In [10], it is shown that $Q_{\alpha\beta}$ can be written in the form (18) with the following substitutions: $\tau \rightarrow -\tau$, $q_m \rightarrow$ action variables θ_α and $p_k \rightarrow$ angle variables ϕ_α canonically conjugated to θ_α .

The important difference of this model from, e.g., the above CM model is that the Hamiltonian (16) and the Lax

matrix (18) are defined not on the whole \mathbf{q} space but only on a subset of it where all $V_j(a, \mathbf{q})$ in (17) are positive (notice the square roots in these expressions). These restrictions depend only on coordinates and on τ (in [10], only the case $0 < \tau < \pi/N$ had been considered). Let $R(\tau, \mathbf{q})$ be the characteristic function of this subset

$$R(\tau, \mathbf{q}) = \begin{cases} 1 & \text{when } V_j(\tau, \mathbf{q}) > 0, j = 1, \dots, N \\ 0 & \text{otherwise} \end{cases} \quad (22)$$

We choose as a “natural” measure for the RS ensemble the uniform measure of random variables \mathbf{p} and \mathbf{q} on the region allowed by the above restrictions. This implies that the measure on the RS ensemble is chosen as

$$dL \sim R(\tau, \mathbf{q}) d\mathbf{p} d\mathbf{q}. \quad (23)$$

We transform this expression to action-angle variables, and perform the integration over angle variables. Since Q and L have the same form but with $\boldsymbol{\lambda}$ and \mathbf{q} interchanged, $\boldsymbol{\lambda}$ is subject to the same restrictions as \mathbf{q} . We conclude that the exact joint probability of eigenvalues of the ensemble of random RS matrices (18) is

$$P(\boldsymbol{\lambda}) \sim R(\tau, \boldsymbol{\lambda}). \quad (24)$$

As mentioned, (18) is a generalization of the model investigated in [18,19]. The simplest nontrivial new case corresponds to the choice $\tau = \pi b/N$ with fixed b . To find $R(\tau, \boldsymbol{\lambda})$ in this case, we notice that as in [18,19], the matrix (18) permits two rank-one deformations with known eigenvectors and eigenvalues $N_{jk}^{(\pm)} = L_{jk} e^{\pm i(q_j - q_k + 2\tau)}$. Generalizing the discussion in [18,19], one can prove [13] that for N large enough, there exist exactly $n = [b]$ other eigenvalues at angular distance $2\pi b/N$ from any eigenvalue θ_α (here $[b]$ denotes the integer part of b).

Consider an ordered sequence of eigenphases on the unit circle, $\theta_1 < \theta_2 < \dots < \theta_N$, and denote the nearest differences by $\xi_k = \theta_{k+1} - \theta_k$. Introducing two functions

$$f(x) = \begin{cases} 1 & \text{when } 0 < x < b \\ 0 & \text{otherwise} \end{cases}, \quad g(x) = 1 - f(x), \quad (25)$$

one can show [13] that these restrictions give rise to the following expression for the joint probability (24) of RS Lax matrix eigenphases inside an interval of length Δ

$$P(\boldsymbol{\xi}) \sim \prod_{j=1}^N f(s_j) g(s_j + \xi_{j+n}) \delta\left(\Delta - \sum_{k=1}^N \xi_k\right), \quad (26)$$

where $s_j = \xi_j + \dots + \xi_{j+n-1}$ and $n = [b]$. This formula means that eigenvalues of RS random matrices (18) with $\tau = \pi b/N$ behave exactly as a 1D gas where each particle interacts with $n = [b]$ nearest neighbors. It is known that in this case, all correlation functions in the limit of large N can be calculated by the transfer operator method (see, e.g., [5]). Here, we present a few results for the k th nearest-neighbor distributions, $P(k, s)$ which determine the probability that in the interval of length s , there exist exactly

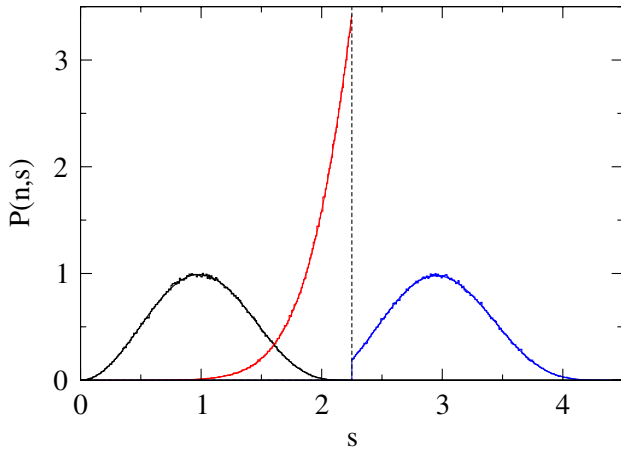


FIG. 2 (color online). Nearest-neighbor distributions (from left to right) $P(1, s)$ (black), $P(2, s)$ (red/gray), and $P(3, s)$ (blue/dark gray) for RS random matrices (29) with $\tau = \pi b/N$ and $b = 9/4$. All curves correspond to $N = 701$ averaged over 1000 realizations of random phases p_k . Solid lines are theoretical predictions. Dashed vertical line indicates abscissa $9/4$.

$k - 1$ other eigenvalues [$P(1, s) \equiv P(s)$]. The details will be discussed elsewhere [13].

When $0 < b < 1$, $P(k, s) = 0$ for $0 < s < kb$ and for $s > kb$ $P(k, s)$ is a shifted Poisson distribution

$$P(k, s) = \frac{e^{-(s-kb)/(1-b)}}{(k-1)!(1-b)^k} (s-kb)^{k-1}. \quad (27)$$

For larger b , formulas, though explicit, become tedious. For example, for $b = 4/3$, $P(1, s)$ is nonzero only when $0 < s < 4/3$ and $P(2, s)$ when $4/3 < s < 8/3$. Inside these intervals

$$P(s) = \frac{81}{64} s^2, \quad P(2, s) = \left(-\frac{3}{2} + \frac{27}{16} s - \frac{81}{512} s^3 \right) e^{3s/4-1}. \quad (28)$$

To simplify numerical calculations, we use [as in (15)] the picket fence approximation of coordinates $q_k = 2\pi k/N$. At these values of \mathbf{q} , $W_j(\tau, \mathbf{q}) = \sin N\tau / (N \sin \tau)$ and the Lax matrix takes the form

$$\tilde{L}_{jk} = e^{ip_k} \frac{1 - e^{2i\tau N}}{N(1 - e^{2\pi i(j-k)/N + 2i\tau})} \quad (29)$$

with random phases p_k uniformly distributed in $[0, 2\pi]$.

When $\tau = \pi\alpha$ with fixed α , this matrix up to notations coincides with the one proposed in [17] and investigated in [18,19]. We put $\tau = \pi b/N$ and perform numerical calculations of spectral statistics of matrix (29) for different values of b and find that above formulas very well agree with numerics. As an illustration, we present at Fig. 2 a case with $b = 9/4$ for which explicit formulas are too long to be presented here. Even in this more complicated case, analytical results are difficult to distinguish from numerics.

Conclusion.—To summarize, we proposed a general method of constructing noninvariant random matrix ensembles whose joint distribution of eigenvalues can be calculated analytically. These ensembles are Lax matrices of classically integrable N -body models, equipped with a suitably chosen measure of momenta and coordinates which depends on the model. For such matrix ensembles, the symmetry groups of usual RMT are replaced by the underlying structure of integrable flows generated by N conserved quantities. It is this structure which makes possible the explicit construction of joint probability of eigenvalues in these ensembles. Spectral statistics of these ensembles are quite unusual, and in many cases they present new examples of nonuniversal intermediate statistics. In all considered cases, eigenfunctions computed numerically present multifractal properties [13], which is a typical feature of intermediate statistics [3]. It is interesting to note that a specific random matrix ensemble, which appeared in [17] as the result of quantization of an interval-exchange map, belongs to this class.

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- [1] M. Berry and M. Tabor, Proc. R. Soc. A **356**, 375 (1977).
 - [2] O. Bohigas, M.-J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).
 - [3] F. Evers and A.D. Mirlin, Rev. Mod. Phys. **80**, 1355 (2008).
 - [4] B.I. Shklovskii *et al.*, Phys. Rev. B **47**, 11487 (1993).
 - [5] E. Bogomolny, U. Gerland, and C. Schmit, Phys. Rev. E **59**, R1315 (1999).
 - [6] P.J. Richens and M.V. Berry, Physica D (Amsterdam) **2**, 495 (1981).
 - [7] M.L. Mehta, *Random Matrix Theory* (Springer, New York, 1990).
 - [8] P. Lax, Commun. Pure Appl. Math. **21**, 467 (1968).
 - [9] S.N.M. Ruijsenaars, Commun. Math. Phys. **115**, 127 (1988).
 - [10] S. Ruijsenaars, Publ. Res. Inst. Math. Sci., Ser. A **31**, 247 (1995).
 - [11] F. Calogero, J. Math. Phys. (N.Y.) **12**, 419 (1971); J. Moser, Adv. Math. **16**, 197 (1975).
 - [12] S.N.M. Ruijsenaars and H. Schneider, Ann. Phys. (N.Y.) **170**, 370 (1986).
 - [13] E. Bogomolny, O. Giraud, and C. Schmit (to be published).
 - [14] M.A. Olshanetsky and A.M. Perelomov, Phys. Rep. **71**, 313 (1981).
 - [15] M. Moshe, H. Neuberger, and B. Shapiro, Phys. Rev. Lett. **73**, 1497 (1994).
 - [16] A.M. García-García and J.J.M. Verbaarschot, Phys. Rev. E **67**, 046104 (2003).
 - [17] O. Giraud, J. Marklof, and S. O'Keefe, J. Phys. A **37**, L303 (2004).
 - [18] E. Bogomolny and C. Schmit, Phys. Rev. Lett. **93**, 254102 (2004).
 - [19] E. Bogomolny, R. Dubertrand, and C. Schmit, Nonlinearity **22**, 2101 (2009).