Shear Viscosity Coefficient and Relaxation Time of Causal Dissipative Hydrodynamics in QCD

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The shear viscosity coefficient and the corresponding relaxation time for causal dissipative hydrodynamics are calculated based on the microscopic formula proposed in T. Koide and T. Kodama [Phys. Rev. E **78**, 051107 (2008)]. Here, the exact formula is transformed into a more compact form and applied to evaluate these transport coefficients in the chiral perturbation theory and perturbative QCD. It is shown that in the leading order calculation, the causal shear viscosity coefficient η reduces to that of the ordinary Green-Kubo-Nakano formula, and the relaxation time τ_π is related to η and pressure P by a simple relationship, $\tau_\pi = \eta/P$.

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The study of dissipative processes in relativistic heavy-ion collisions is now one of the important topics to clarify quantitatively how closely the matter created there behaves as an ideal fluid. The proper concept of relativistic dissipative hydrodynamics is, however, not trivial at all. It is by now known that a naive covariant extension of the Navier-Stokes equation leads to the problem of relativistic acausality and instabilities [1].

An essential key to this question is the presence of memory effect characterized by a finite relaxation time in the definition of irreversible currents [2]. Consequently, in a relativistic regime, any fluid becomes *non-Newtonian*, that is, the irreversible current is not simply proportional to thermodynamic forces. We refer to those theories which incorporate this effect as the causal dissipative hydrodynamics (CDH).

Similarly to equation of states, transport coefficients reflect the properties of the matter. These are inputs for hydrodynamic modelings and should be determined from a microscopic theory. So far, there are mainly two approaches to estimate these coefficients of CDH. One is the kinetic approach based on the Boltzmann equation [3,4]. The other is the calculation from a duality assuming the AdS/CFT (Anti-de Sitter space/ conformal field theory) correspondence [5,6]. However, the applicability of these approaches is not obvious for the physics of heavy-ion collisions. The kinetic approach is only applicable for rarefied gases where the Boltzmann-Grad limit is satisfied. The AdS/CFT approach predicts the behavior of a matter described by a conformal field theory in its the strong coupling limit, but QCD is obviously not a conformal theory.

The Green-Kubo-Nakano (GKN) formula is another possibility which is used to evaluate these transport coefficients in terms of microscopic correlation functions. However, for our purpose, there exist the following problems. One is that, in this approach, the relaxation time has not been expressed with correlation functions. Furthermore, it is not obvious whether the GKN formula for the shear viscosity coefficient is applicable when the system exhibits a non-Newtonian nature of irreversible currents. Usually, Newtonian nature is assumed in the derivation of the GKN formula but, as mentioned above, this is exactly the key question for the case of a relativistic fluid [7,8]. More precisely, the shear viscosity coefficient appears as a phenomenological parameter, that is, the diffusion constant for transverse momentum via a ratio to entropy density s in the phenomenological Langevin approach. However, this proportionality to the entropy density is not always true and depends on its definition from a microscopic theory.

Another important point is that the hydrodynamic shear viscosity, for example, appears in a thermal relaxation process which is not a response of the system to an external force, but is rather a response to the inhomogeneity of the velocity field, that is, the change in boundary conditions. Thus, a direct application of the traditional argument of the linear response theory for the calculation of the shear viscosity is not straightforward. As a matter of fact, the GKN formula of the shear viscosity is derived by using, for example, the nonequilibrium statistical operator method [9].

Recently, the microscopic formulae to calculate the transport coefficients of CDH was derived by use of the projection operator method [7,8,10]. Although this approach is shown to be very powerful, the results obtained so far are yet too formal for the practical applications. The aim of the present work is twofold: one is to reduce the formula obtained in [7,8] into a more compact form, and

the other is to apply it to the hadronic matter and quark-gluon plasma (QGP) to investigate the temperature dependence of the causal shear viscosity coefficient η and the relaxation time τ_π.

So far, τ_π has not been discussed enough compared to η, but some crucial questions depend on the precise behavior of the relaxation time. For example, the behavior of τ_π is directly related to the stability of relativistic fluids as discussed in [1]. Another question is the value of η/s obtained form the analysis of experimental data seems very small, as close, if not smaller than, to the lower bound $1/4\pi$ predicted by the AdS/CFT correspondence. However, even if η/s is not so small, the fluid can behave as an ideal fluid when τ_π is sufficiently large. Thus, to know the precise value of τ_π is fundamental.

For simplicity, let us consider a system with shear flow, where the fluid velocity points to the x direction with finite velocity gradient in the y direction. We further assume that the evolution equation of the shear viscosity is approximately given by the evolution of the two variables, T^{yx} and T^{0x}. Correspondingly, we will employ the formula of $N = 2$ form given in Sec. VIII of [8], where N is the dimension of the projected space. The exact time evolution of spatial, off-diagonal components of energy-momentum tensor $T^{\mu\nu}(\mathbf{k}, t)$ in the momentum representation (Fourier transform) is formally expressed as

$$\partial_t T^{yx}(k_y, t) = -ik_y R_{k_y} T^{0x}(k_y, t) - \int_0^t d\tau \Xi_{22}(k_y, \tau) T^{yx}(k_y, t-\tau) + \xi_{k_y}(t), \tag{1}$$

where $R_{k_y} = (T^{yx}(k_y), T^{yx}(-k_y))/(T^{0x}(k_y), T^{0x}(-k_y))$, and the inner product (X, Y) represents Kubo's canonical correlation $(X, Y) = \int_0^\beta d\lambda \beta^{-1} \mathrm{Tr}[\rho_{\mathrm{eq}} e^{\lambda H} X e^{-\lambda H} Y]$. Here, H is the Hamiltonian, and β is the inverse of temperature $1/T$. The memory function $\Xi_{22}(k_y, t)$ is defined in Eq. (78) of [8]. The last term of the Eq. (1) is often called the noise term, and related to the memory term through the fluctuation-dissipation theorem of the second kind [7,8].

To break the time-reversal symmetry, we introduce the time-convolutionless (TCL) approximation [7,8],

$$\partial_t T^{yx}(k_y, t) = -ik_y(\varepsilon + P) R_{k_y} u^x(k_y, t) - \frac{1}{\tau_{k_y}} T^{yx}(k_y, t), \tag{2}$$

where we have introduced a function related to the relaxation time $\tau_{k_y}^{-1} = \int_0^\infty d\tau \Xi_{22}(k_y, \tau)$, and ε, P, and u are the energy density, pressure, and fluid velocity, respectively. The noise term is not necessary for the present discussion and dropped out. The TCL approximation is equivalent to assume the exponential ansatz for the memory of $T^{yx}(k_y, t)$. It should be emphasized that, even after the TCL approximation, the hysteresis of T^{xy} is still reserved in its time evolution equation, and this corresponds exactly to the memory effect in the usual CDH [2]. In this derivation, we have used the following replacement $T^{0x} = (\varepsilon + P) u^x(k_y, t)$, which is justified near the local rest frame.

On the other hand, the linearized phenomenological equation for the shear viscosity defined by projection to the traceless part, $\pi^{\mu\nu} = \frac{1}{2}(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha} - \frac{2}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}) T_{\alpha\beta}$ with $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$, is given by

$$\tau_\pi \frac{\partial}{\partial t} \pi^{yx}(k_y) + \pi^{yx}(k_y) = -\eta(ik_y) u^x(k_y) \tag{3}$$

at the local rest frame. Comparing the above equation with Eq. (2), we identify the η and τ_π [8]:

$$\tau_\pi = \lim_{k_y \to 0} \tau_{k_y}, \qquad \eta = \lim_{k_y \to 0} (\varepsilon + P) R_{k_y} \tau_{k_y}. \tag{4}$$

The memory function can be cast into a simple form in the low momentum limit:

$$\Xi_{22}^L(k_y, s) = \frac{1 - s X_{22}^L(k_y, s)}{X_{22}^L(k_y, s)} + O(k_y), \tag{5}$$

where $X_{22}^L(k_y, s)$ is the Laplace transform with respect to t of the following correlation function,

$$X_{22}(k_y, t) = \frac{(T^{yx}(k_y, t), T^{yx}(-k_y, 0))}{(T^{yx}(k_y, 0), T^{yx}(-k_y, 0))}. \tag{6}$$

Substituting this equation into the definitions (4), we obtain $\tau_\pi = X_{22}^L(\mathbf{0}, 0)$ and $\eta = (\varepsilon + P) R_{\mathbf{0}} X_{22}^L(\mathbf{0}, 0)$.

For later convenience, we re-express the correlation function $X_{22}^L(\mathbf{0}, 0)$ with the definition of the GKN shear viscosity coefficient [9] as

$$X_{22}^L(\mathbf{0}, 0) = \frac{\eta_{\mathrm{GKN}}}{\beta} \left[\int d^3\mathbf{x} (\pi^{xy}(\mathbf{x}, 0), \pi^{xy}(\mathbf{0}, 0)) \right]^{-1}, \tag{7}$$

where

$$\eta_{\mathrm{GKN}} = -\frac{1}{10} \int d^3\mathbf{x} \int_{-\infty}^t dt_1 \times \int_{-\infty}^{t_1} d\tau \langle \pi^{\alpha\beta}(\mathbf{x}, t) \pi_{\alpha\beta}(\mathbf{0}, \tau) \rangle_{\mathrm{ret}} (\alpha, \beta = 0, 1, 2, 3). \tag{8}$$

Here the suffix ret denotes the retarded Green function, and there is difference by factor 2 from [9] because of the difference of the definition.

Finally, η and τ_π are given by

$$\frac{\eta}{\beta(\varepsilon + P)} = \frac{\eta_{\mathrm{GKN}}}{\beta^2 \int d^3\mathbf{x} (J^x(\mathbf{x}, 0), J^x(\mathbf{0}, 0))}, \tag{9}$$

$$\frac{\tau_\pi}{\beta} = \frac{\eta_{\mathrm{GKN}}}{\beta^2 \int d^3\mathbf{x} (\pi^{xy}(\mathbf{x}, 0), \pi^{xy}(\mathbf{0}, 0))}, \tag{10}$$

where we have introduced the energy current, $J^\mu(\mathbf{x}, t) = u_\nu T^{\nu\mu}(\mathbf{x}, t)$ which is $T^{0\mu}(\mathbf{x}, t)$ at the local rest frame. Note that $\eta/[\beta(\varepsilon + P)]$ is nothing but η/s at vanishing chemical potential. These expressions are still most general ones

within a framework where the hydrodynamic description is meaningful and are the first main results of the present work.

Concerning the causality problem in CDH, we introduce a ratio $\eta/[\tau_\pi(\varepsilon + P)]$, which is, in our case, given by R_0 itself. Note that the propagation speed of signal should be slower than the speed of light. For the $(3 + 1)$-dimensional viscous fluid to be causal in the shear channel, the following relation should be satisfied [1],

$$\frac{\eta}{\tau_\pi(\varepsilon + P)} \leq \frac{3}{4}(1 - c_s^2), \tag{11}$$

where c_s is the ordinary sound velocity $c_s^2 = dP/d\varepsilon$. In particular, c_s^2 is $1/3$ in the massless ideal gas case, and this $\eta - \tau_\pi$ ration should be smaller than $1/2$.

Now we apply our formula to QCD-related problems. In the hadronic phase at low baryon density and high temperature, the dominant behavior comes from the pions. The interaction of pions can well be described by the chiral perturbation theory which is a low energy effective theory of QCD. To estimate the GKN shear viscosity coefficient η_{GKN}, we adapt the leading order result given in [11]. At the high temperature regime far above the deconfinement scale, we assume that the dominant behavior of the QGP is given by gluon, and use η_{GKN} obtained to the leading order in the perturbative QCD (pQCD) calculation for $N_f = 0$ in [12]. We take the running coupling constant to have the same parametrization as [13].

In order to estimate η and τ_π to the leading order in couplings, it is enough to calculate the fluctuations of J^μ and $\pi^{\mu\nu}$ to the lowest order. Then, we obtain

$$\int d^3\mathbf{x}(J^x(\mathbf{x}), J^x(\mathbf{0})) = \frac{g}{3}\int \frac{d^3p}{(2\pi)^3}\mathbf{p}^2 n_p(1 + n_p) = \frac{\varepsilon + P}{\beta}, \tag{12}$$

$$\int d^3\mathbf{x}(\pi^{xy}(\mathbf{x}), \pi^{xy}(\mathbf{0})) = \frac{g}{15}\int \frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}^2)^2}{\beta E_p^2} \times \left(\frac{1}{E_p} + \beta(1 + n_p)\right) n_p = \frac{P}{\beta}, \tag{13}$$

where $n_p = (e^{\beta E_p} - 1)^{-1}$ is the Bose-Einstein distribution function. The statistical factor g is 3 for pions and 16 for gluons. In the derivation of Eq. (13), we have subtracted the vacuum contribution. Here, ε and P are the energy density and pressure of the ideal Bose gas, respectively.

By substituting them into the definitions Eqs. (9) and (10), we obtain the leading order results. In this approximation, we found that the causal shear viscosity coefficient is given by $\eta = \eta_{\mathrm{GKN}}$, and the relaxation time (10) reduces to

$$\tau_\pi = \eta_{\mathrm{GKN}}/P = \eta/P. \tag{14}$$

See [11,13,14] for the behavior of η_{GKN}.

We show the temperature dependence of τ_π to β ratio in Fig. 1. The behavior is very similar to that of η/s: a decreasing function of temperature in the hadron phase and an increasing one in the QGP phase, exhibiting minimum near the phase transition temperature T_c. The dashed line denotes the prediction from the AdS/CFT correspondence $\tau_\pi/\beta = (2 - \ln 2)/(2\pi)$ for $\mathcal{N} = 4$ SYM [6]. However, because of the weak temperature dependence of τ_π/β in the QGP phase, τ_π itself is a decreasing function for both of the hadron and QGP phases and show discontinuity near T_c, that is, τ_π becomes minimum (maximum) in the hadron (QGP) side near T_c. Thus, the behavior of the shear viscosity is insensitive for the rapid change of the fluid velocity above T_c because of the large τ_π; meanwhile, it is more sensitive below T_c. In this sense, the matter created in relativistic heavy-ion collisions can be close to the ideal fluid in two different ways. One is because of the small η and τ_π which may be realized in the hadron phase, and the other is the small η but large τ_π in the QGP phase.

The relaxation time is calculated also from the relativistic Boltzmann equation using Grad's moment method with the 14-moment approximation [4,15]. In order to compare with our result, the temperature dependence of τ_π to η ratio in the unit of pressure P is shown in Fig. 2. The solid line denotes the behavior of our result, which should be one in our leading order calculation. The dot-dashed line denotes the result from the AdS/CFT correspondence, $(2 - \ln 2)/2$ [6]. The dashed line represents the result obtained from the moment method for the pion (Bose-Einstein) gas without phase transition [4]. This figure shows that our theory gives a different result from the 14-moment calculation. In the Navier-Stokes limit (Newtonian fluid), it is known that the momentum method with the 13-moment approximation and the famous Chapman-Enskog procedure are consistent for the shear viscosity. However, τ_π requires expansions in higher moments and, to see the relation of our formula and the

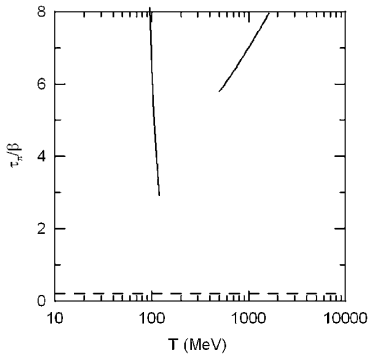

FIG. 1. The temperature dependence of the relaxation time to β ratio with $\beta = 1/T$. The solid and dashed lines represent our result and that from AdS/CFT correspondence, respectively.

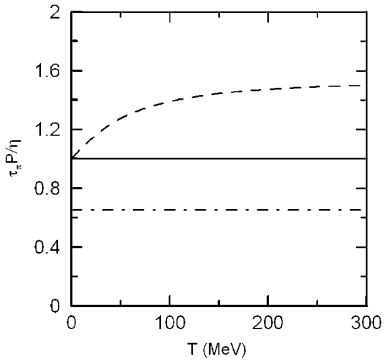

FIG. 2. The temperature dependence of $\tau_\pi P/\eta$. The solid, dashed and dot-dashed lines correspond to the result of our formula, the moment method and AdS/CFT correspondence, respectively.

moment method, a more careful comparison should be done. It is also worth mentioning that there is a attempt to calculate these coefficients without using the moment method [16].

As the AdS/CFT correspondence predicts that the minimum of $\eta_{\rm GKN}/s$ has a lower bound $1/(4\pi)$ [5], our η/s and τ_π would have a lower bound, unless the fluctuation of $\pi^{\mu\nu}$ diverges. As a matter of fact, to be consistent with the causality condition (11), τ_π is somehow correlated with η, and τ_π/β cannot be much smaller than $\eta/[\beta(\varepsilon + P)]$. In Fig. 3, we show the temperature dependence of the η-τ_π ratio. In our leading order calculation, this ratio is nothing but $\frac{\eta}{\tau_\pi(\varepsilon+P)} = \frac{P}{(\varepsilon+P)}$, and shows a nontrivial temperature dependence only at lower temperature than the pion mass and finally converges to the massless ideal gas limit $1/4$ where $\varepsilon = 3P$. One can easily see that the result satisfies the relativistic causality condition (11). It should be emphasized that relativistic fluids become unstable if the causality condition is not satisfied [1]. To see the consistency of CDH, it is necessary to investigate this ratio more carefully.

In summary, we proposed the compact definitions of the transport coefficients of CDH, and calculated them in the chiral perturbation theory and pQCD. We found that, in the leading order calculation, the causal shear viscosity coefficient η is reduced to that of the GKN shear viscosity coefficient $\eta_{\rm GKN}$. Although intuitive, this result is not trivial *a priori* because the irreversible current is not simply proportional to the thermodynamic force in CDH. In fact, τ_π does not vanish in our calculation, and there is a simple relationship, $\tau_\pi = \eta/P$. This relation is not the same as that obtained in the 14-moment calculation, and the physical origin of this discrepancy should be clarified. So far, the improvements of the moment method has been discussed in diverse ways [17]. Our result may give a milestone in such a development.

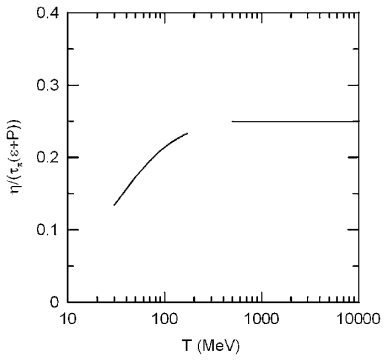

FIG. 3. The temperature dependence of the η-τ_π ratio.

It is thus very interesting to ask how these are modified when the calculation is implemented beyond the leading order. The simple relationship obtained here would not be satisfied in the strong coupling limit. In fact, from AdS/CFT results, we can show $\eta/\tau_\pi = Ts/(4 - 2\ln 2)$, which is larger than the equation of state of the massless ideal fluid, $P = Ts/4$, that is, $P \neq \eta/\tau_\pi$ as was shown in Fig. 2. Thus, we expect that the simple relation $\eta = \eta_{\rm GKN}$ and/or $\tau_\pi = \eta/P$ calculated from the exact formulae (9) and (10) would acquire a nontrivial temperature dependence once going beyond the leading order result and in the strong coupling limit.

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