

## Measuring a Long-Range Dark Matter Force at the Large Hadron Collider

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A long-range “dark force” has recently been proposed to mediate the dark matter (DM) annihilation. If DM particles are copiously produced at the Large Hadron Collider, the light dark force mediator will also be produced through radiation. We demonstrate how and how precise we can utilize this fact to measure the coupling constant of the dark force. The light mediator’s mass is measured for the “lepton jet” to which it decays. In addition, the mass of the DM particle is determined using the  $m_{T2}$  technique. Knowing these quantities is critical for calculating the DM relic density.

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**Introduction.**—Recently, there have been strong indications of indirect detection of dark matter (DM) particles. In particular, the payload for antimatter matter exploration and light-nuclei astrophysics (PAMELA) experiment has observed a sharp turnover of positron fraction in the cosmic rays in the 10–100 GeV range [1], which can be naturally explained by DM-DM annihilation to electron-positron pairs with a large “boost factor.” In an interesting DM scenario [2], the boost factor is obtained through the Sommerfeld enhancement effect [3], the consequence of a long-range attractive force between DM particles. A light mediator  $a_{\text{dark}}$  with an  $O(\text{GeV})$  mass provides the attractive force. Given its small mass,  $a_{\text{dark}}$  must couple to the standard model (SM) very weakly (hence the force it mediates is given the name the “dark force”); otherwise, its effects would have been seen long ago. Accordingly, its coupling to DM has to be of the order of unity to obtain the required annihilation rate. The lack of antiproton excess in the PAMELA experiment [4] can be explained if  $a_{\text{dark}}$  is so light that its decays to hadrons are kinematically forbidden [2] or if it dominantly couples to the SM leptons [5].

The above DM scenario also has unusual signatures at the LHC [6]. If  $a_{\text{dark}}$  decays within a collider detector and dominantly to leptons, we will be able to see “lepton jets,” where two or more leptons are boosted and collinear to one another. Once events with lepton jets as well as large missing energy are observed, it is critical to test whether they come from the same theory that explains the positron/electron anomalies. The purpose of this Letter is to discuss how to extract the relevant information from the LHC measurements to test this new class of DM models. In particular, we demonstrate how to measure the DM particle mass, the light mediator mass, and the coupling constant  $g$  of the DM- $a_{\text{dark}}$  interaction. Once these quantities are determined, we will be able to calculate the DM-DM annihilation rate. This allows us to compare with the results of DM searches as well as to verify if we obtain the correct DM relic density, assuming it is thermally produced.

To be able to perform these measurements, it is crucial that the DM particle can be produced at the LHC, which we assume to be the case. Once it is produced, it has a significant probability to radiate an extra particle  $a_{\text{dark}}$  (Fig. 1), because  $a_{\text{dark}}$  is light and the DM- $a_{\text{dark}}$  coupling is large. Therefore, for whatever process to produce the DM particle, there is a corresponding process with an extra  $a_{\text{dark}}$  produced through final state radiation. The production rate ratio of the two processes in general depends on  $M_{a_{\text{dark}}}$ , the coupling  $g$ , and the DM mass  $m_{\text{DM}}$ , which can be written as

$$\frac{\sigma(pp \rightarrow XDMDM a_{\text{dark}})}{\sigma(pp \rightarrow XDMDM)} \approx C \frac{g^2}{4\pi^2} \log\left(\frac{q^2}{M_{a_{\text{dark}}}^2}\right) \log\left(\frac{q^2}{m_{\text{DM}}^2}\right). \quad (1)$$

Here  $X$  represents visible particles;  $C$  is a process-dependent coefficient. This  $q^2$ -dependent result is known as the Sudakov double logarithm. If other particles charged under the dark force are produced, they can also radiate an  $a_{\text{dark}}$  in which case  $m_{\text{DM}}$  in Eq. (1) should be replaced by the relevant mass. As we will show shortly,  $M_{a_{\text{dark}}}$  and  $m_{\text{DM}}$  can be determined independently, which enables us to further extract the coupling.

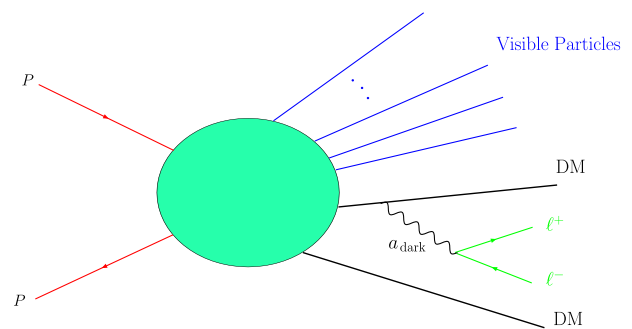


FIG. 1 (color online). The schematic Feynman diagram of the dark matter radiating a light mediator  $a_{\text{dark}}$ , which decays to two leptons.

To be concrete, in this Letter we envisage that the LHC signal events are produced in a two-step process. The first step is entirely analogous to ordinary models with missing particles, such as the minimal supersymmetric standard model (MSSM) with  $R$  parity or the universal extra dimension with  $KK$  parity. We will adopt the nomenclature of the MSSM although a general setup is understood. Large cross sections are achieved when colored particles are produced and subsequently decay to the lightest supersymmetric particle (LSP) in the MSSM. Because of the small coupling between the DM and the SM, these decays are not affected. In addition, we assume the MSSM LSP is heavier than the DM, which is  $R$  parity odd. Therefore, in the second step, the MSSM LSP decays to the DM plus some extra visible particles. In the following, we exemplify our method by analyzing a simple model in the above scenario. After giving the essential ingredients of the model, we discuss in turn the measurements of the mediator mass, the DM mass, and the coupling. We conclude the Letter with a few discussions.

*The measurements.*—The visible sector in our model is the ordinary MSSM with a binolike neutralino LSP  $\tilde{\chi}_1^0$ . The dark sector contains a  $U(1)_{\text{dark}}$  gauge group and two Higgs superfields with opposite charges under  $U(1)_{\text{dark}}$ . The dark sector interacts with the ordinary MSSM sector through a small kinetic mixing between the  $U(1)_{\text{dark}}$  gauge superfield and the  $U(1)_Y$  gauge superfield. We assume that the dark sector LSP is the dark Higgsino  $\tilde{\chi}_{\text{dark}}$ , which is our DM candidate. Correspondingly, the dark gauge boson, which is identified as the mediator  $a_{\text{dark}}$ , provides an attractive force between two  $\tilde{\chi}_{\text{dark}}$ 's.  $U(1)_{\text{dark}}$  is broken by the vacuum expectation values of the dark Higgs fields (the lightest physical Higgs boson is denoted as  $h_{\text{dark}}$ ), which provide  $a_{\text{dark}}$  a mass of  $O(1 \text{ GeV})$ . Because of the kinetic mixing,  $a_{\text{dark}}$  decays to two SM leptons, which are collinear with each other and form a “dark gauge boson jet” (or  $a$  jet). For the dark Higgs boson  $h_{\text{dark}}$ , we assume it has a mass  $M_{h_{\text{dark}}} \geq 2M_{a_{\text{dark}}}$ . Therefore it mainly decays to two  $a_{\text{dark}}$ 's which subsequently decay to four leptons forming a “dark Higgs jet” (or  $h$  jet). Given the assumption that  $\tilde{\chi}_{\text{dark}}$  is lighter than  $\tilde{\chi}_1^0$ ,  $\tilde{\chi}_1^0$  mainly decays to  $\tilde{\chi}_{\text{dark}}$  plus  $h_{\text{dark}}$  or  $a_{\text{dark}}$ .  $\tilde{\chi}_1^0$  can also undergo a three-body decay to  $\tilde{\chi}_{\text{dark}} + h_{\text{dark}} + a_{\text{dark}}$ . The ratio of the three-body and the two-body decay widths can be measured and used to determine the coupling  $g$ . For this purpose, it is enough to count the number of events containing two  $h$  jets and the number of events containing two  $h$  jets plus one  $a$  jet.

Given the above setup, we choose the dark matter mass to be 600 GeV, consistent with the ATIC results. We also fix  $M_{a_{\text{dark}}} = 1 \text{ GeV}$  and  $M_{h_{\text{dark}}} = 3 \text{ GeV}$ . The coupling constant  $g$  is 0.40 to provide the correct DM relic density, which determines the boost factor to be  $O(100)$ . In the MSSM sector, we choose the masses of the gluino, squarks (restricted to the first two generations), and the MSSM LSP  $\tilde{\chi}_1^0$  to be 1200, 1000, and 700 GeV, respectively. The spectrum is chosen such that the gluino directly decays to

quark plus squark and the squark only directly decays to quark plus  $\tilde{\chi}_1^0$ . We generate the parton level events in the squark (gluino) pair production channels with MADGRAPH (MADEVENTS) [7] for the LHC at 14 TeV. The total cross section is 0.84 pb. The two-body and three-body decays for  $\tilde{\chi}_1^0$  are performed with CALCHEP [8], and all other particles, including  $a_{\text{dark}}$ ,  $h_{\text{dark}}$ , and the other superparticles, are decayed with BRIDGE [9]. Here we assume that  $a_{\text{dark}}$  100% decays to two muons, and we will comment on the case with  $a_{\text{dark}}$  decaying to electrons later. The parton level events are further processed with PYTHIA [10] for showering/hadronization and PGS [11] for detector simulation.

The lepton jets are reconstructed as follows: All muons are first sorted according to their  $p_T$ 's. Then we choose the muon with the highest  $p_T$  in the list as a “seed” for the lepton jet and add muons within 0.2 rad of the seed muon direction to the lepton jet. Used muons are removed from the list. The “seed axis” is fixed in this procedure. We repeat this procedure until all muons are used. Lepton jets with 2 muons are tagged as  $a$  jets, and lepton jets with 3 or 4 muons are tagged as  $h$  jets. Events containing untagged muons are discarded. An  $H_T > 500 \text{ GeV}$  cut is imposed to reduce the SM background, where  $H_T$  is defined as the scalar sum of all objects'  $p_T$  (including  $\cancel{p}_T$ ) in the events. We list the numbers of background and signal events after cuts in Table I. The SM backgrounds coming mainly from  $b\bar{b}$  and  $t\bar{t}$  final states are negligibly small after imposing a mass window cut on  $a$  jets.

*Measuring  $M_{a_{\text{dark}}}$  and  $M_{h_{\text{dark}}}$ .*—The mass measurements of the particles  $a_{\text{dark}}$  and  $h_{\text{dark}}$  are straightforward since that they are given, respectively, by the invariant masses of the  $a$  jet and the  $h$  jet. The precision of the mass measurements is therefore determined by the precision of muon momentum measurements. We simulate the toroidal LHC apparatus (ATLAS) detector resolution with ATLFast [12], which smears both the magnitudes and the directions of the muon momenta, and estimate the mass measurement error as  $\sim M/(100\sqrt{N})$ , where  $M$  denotes  $M_{a_{\text{dark}}}$  or  $M_{h_{\text{dark}}}$  and  $N$  is the number of corresponding lepton jets. This precision is much better than the other measurements, as discussed below.

*Measuring  $m_{\tilde{\chi}_1^0}$  and  $m_{\tilde{\chi}_{\text{dark}}}$ .*—To determine the masses of  $\tilde{\chi}_1^0$  and  $\tilde{\chi}_{\text{dark}}$ , we adopt a method based on the variable  $m_{T2}$  [13] on events with 2  $h$  or  $a$  jets. This is an inclusive method since that  $\tilde{\chi}_1^0$  decays to  $\tilde{\chi}_{\text{dark}}$  in both decay chains

TABLE I. The numbers of background and signal events after cuts. A  $10 \text{ fb}^{-1}$  luminosity and a 14 TeV center of mass energy are assumed for the LHC. The final signal events after cuts contain 5939 two-body events and 31  $2h1a$  three-body events.

	$b\bar{b}$	$t\bar{t}$	$W/Z$ 's	Signal
$H_T > 500 \text{ GeV}$	1106 k	1068 k	17 k	8303
No. of muons $\geq 4$	168	1890	13	7650
No. of lepton jets $\geq 2$	70	36	<1	6523
$ M_{a_{\text{jet}}} - 1 \text{ GeV}  \leq 0.1 \text{ MeV}$	1	1	<1	5970

in every event. Alternatively, one could consider more specific processes and methods using more kinematic information [14]. For  $10 \text{ fb}^{-1}$ , we obtain 5941 events with 2  $h/a$  jets after the cuts in Table I. We use  $\mu_{\tilde{\chi}_{\text{dark}}}$  and  $\mu_{\tilde{\chi}_1^0}$  to denote some trial masses for  $\tilde{\chi}_1^0$  and  $\tilde{\chi}_{\text{dark}}$  and examine the number of events that are kinematically consistent with each  $(\mu_{\tilde{\chi}_{\text{dark}}}, \mu_{\tilde{\chi}_1^0})$ . As discussed in Ref. [15],  $(\mu_{\tilde{\chi}_{\text{dark}}}, \mu_{\tilde{\chi}_1^0})$  is consistent with a given event if and only if  $\mu_{\tilde{\chi}_1^0} > m_{T2}(\mu_{\tilde{\chi}_{\text{dark}}})$ , where  $m_{T2}$  is calculated using  $\not{p}_T$  and the momenta of the two lepton jets. Using this fact, we easily assign the number of consistent events for each point on the  $(\mu_{\tilde{\chi}_{\text{dark}}}, \mu_{\tilde{\chi}_1^0})$  mass plane (Fig. 2).

The top contour in Fig. 2 corresponds to the  $m_{T2}$  end points  $m_{T2}^{\text{max}}(\mu_{\tilde{\chi}_{\text{dark}}})$ . We note that the  $m_{T2}^{\text{max}}$  contour ‘‘curls up’’ on both ends with respect to a straight line with constant mass difference and tangent to the contour. This is caused by the presence of upstream transverse momentum provided by the squark or gluino decay as well as the initial state radiation [16]. We count the number of consistent events along this line, which is maximized around  $\mu_{\tilde{\chi}_{\text{dark}}} = m_{\tilde{\chi}_{\text{dark}}}$  (Fig. 3). We fit the number of events distribution with a 5th order polynomial and take the maximum position as our estimation of the DM mass. The error of this measurement is estimated by repeating the procedure for 10 different data sets and computing the deviation from the average value, which gives us  $m_{\tilde{\chi}_{\text{dark}}} = 613 \pm 12 \text{ GeV}$  (the input value is 600 GeV) and  $m_{\tilde{\chi}_1^0} - m_{\tilde{\chi}_{\text{dark}}} = 101.9 \pm 0.8 \text{ GeV}$  (the input value is 100 GeV).

*Measuring the coupling constant.*—As mentioned, we are interested in the ratio of effective cross sections of  $2h_{\text{dark}}$  and  $2h_{\text{dark}}1a_{\text{dark}}$  events. Taking the combinatorial factor into account, this ratio is equal to twice the ratio  $R = \text{BR}(\tilde{\chi}_1^0 \rightarrow \tilde{\chi}_{\text{dark}} h_{\text{dark}} a_{\text{dark}}) / \text{BR}(\tilde{\chi}_1^0 \rightarrow \tilde{\chi}_{\text{dark}} h_{\text{dark}})$ . For a fixed ratio of  $M_{h_{\text{dark}}} / M_{a_{\text{dark}}} = 3$ ,  $M_{a_{\text{dark}}} \ll m_{\tilde{\chi}_1^0} - m_{\tilde{\chi}_{\text{dark}}}$ , and  $m_{\tilde{\chi}_1^0} - m_{\tilde{\chi}_{\text{dark}}} \ll m_{\tilde{\chi}_1^0}$ , we obtain an approximate formula

$$R \approx \frac{11g^2}{120\pi^2} [x^2 - (8\log 2 + 4)x + 4(3\log^2 2 + 4\log 2 + 2)], \quad (2)$$

with  $x \equiv 2 \log[(m_{\tilde{\chi}_1^0}^2 - m_{\tilde{\chi}_{\text{dark}}}^2) / (M_{a_{\text{dark}}} m_{\tilde{\chi}_1^0})]$ .

Note that, when  $M_{a_{\text{dark}}}$  is significantly below 1 GeV, this ratio  $R$  becomes very large and one should include higher order processes with more  $a_{\text{dark}}$  radiated. We also numerically show the ratio  $R$  as a function of  $M_{a_{\text{dark}}}$  for a few different values of  $M_{h_{\text{dark}}} / M_{a_{\text{dark}}}$  in Fig. 4. As can be seen from Fig. 4, the approximate formula in Eq. (2) fits the numerical results very well and will be used for error analysis of the gauge coupling measurement. The double logarithm dependence on  $r_a$  or  $M_{a_{\text{dark}}}$  is due to the fact that the radiated  $a_{\text{dark}}$  tends to be collimated with  $h_{\text{dark}}$ .

In our model, the annihilation rate for DM to two  $a_{\text{dark}}$ 's or 1  $a_{\text{dark}}$  plus 1  $h_{\text{dark}}$  is given by the  $S$ -wave result (for a heavy dark gaugino)

$$\sigma v = \left(\frac{g}{0.40}\right)^4 \left(\frac{m_{\tilde{\chi}_{\text{dark}}}}{600 \text{ GeV}}\right)^{-2} \times 2.31 \times 10^{-26} \text{ cm}^3/\text{s}, \quad (3)$$

where the required rate  $2.31 \pm 0.07 \times 10^{-26} \text{ cm}^3/\text{s}$  is inferred [17] from the WMAP result  $\Omega_{\text{dm}} h^2 = 0.113 \pm 0.0034$  [18]. An 8% increase in the annihilation cross section from the Sommerfeld enhancement is taken into account in Eq. (3). Hence, we have chosen  $g = 0.40$  and  $m_{\tilde{\chi}_{\text{dark}}} = 600 \text{ GeV}$  to give the correct DM relic density. Then  $R \approx 0.052$  for  $M_{a_{\text{dark}}} = 1 \text{ GeV}$ . As stated before, this ratio can be measured by counting the number of events with 2  $h$  jets and the number of events with 2  $h$  jets plus 1  $a$  jet. The latter number is much smaller, which dominates the error for the  $R$  measurement. For  $10 \text{ fb}^{-1}$ , we expect 220  $2h1a$  events before any cuts. However, the  $a$  jets tend to be collinear with the  $h$  jets and/or contain soft muons that are not registered by the detector. This drastically reduces the efficiency for identifying  $2h1a$  events to about 14% [19], which has to be taken into consideration when calculating the ratio  $R$ . The number of events is reduced to 31 after lepton-jet reconstruction, which results in an error about 18% for the measurement of the ratio  $R$ . Alternatively, we can treat a lepton jet with 5 or 6 muons as collinear  $h_{\text{dark}} + a_{\text{dark}}$  from  $\tilde{\chi}_1^0$  three-body decay, which

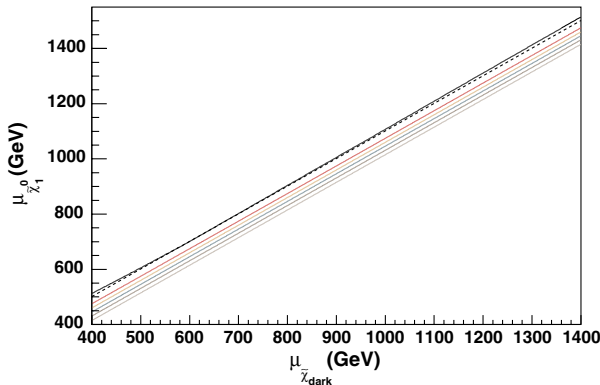


FIG. 2 (color online). Contour plot of the number of events consistent with the mass pair  $(\mu_{\tilde{\chi}_{\text{dark}}}, \mu_{\tilde{\chi}_1^0})$ . Starting from the top contour, the number of events descends by 1000 from the adjacent contour. Above the top contour, the masses are consistent with all 5941 events. The dashed straight line has fixed mass difference, which is tangent to the top contour.

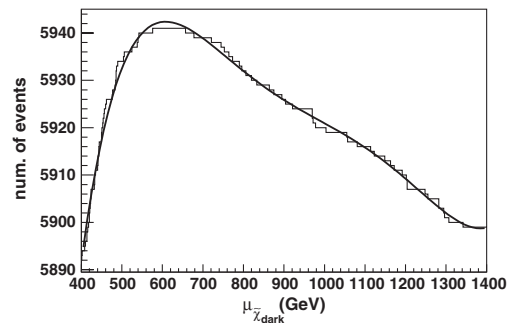


FIG. 3. Number of consistent events along the line with fixed mass difference and tangent to the  $m_{T2}^{\text{max}}$  contour.

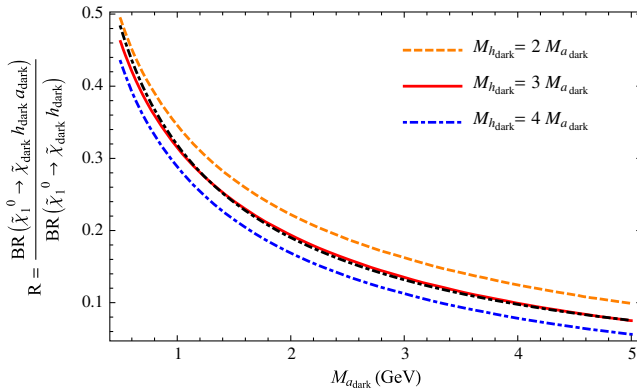


FIG. 4 (color online). The ratio  $R$  as a function of  $M_{a, \text{dark}}$  for different values of  $M_{h, \text{dark}}/M_{a, \text{dark}}$  and fixed values  $m_{\tilde{\chi}_1^0} = 700$  GeV and  $m_{\tilde{\chi}_{\text{dark}}} = 600$  GeV. The ratios are proportional to the gauge coupling square, which is chosen to be  $g = 1$  here. The black dashed line is calculated using the approximate formula in Eq. (2).

then increases the number of events to 70 and reduces the error to 12%.

Having measured the masses and the ratio  $R$ , we determine the dark gauge coupling measurement from Eq. (2) as  $g = 0.41 \pm 0.03$ . The error for  $g$  is obtained by summing in quadrature the errors of all parameters on which it depends. Since  $M_{a, \text{dark}}$ ,  $M_{h, \text{dark}}$ , and  $m_{\tilde{\chi}_1^0} - m_{\tilde{\chi}_{\text{dark}}}$  can be measured relatively more precisely, the error is dominated by the errors on  $R$  and  $m_{\tilde{\chi}_{\text{dark}}}$ . Using Eq. (3), we conclude that, from the LHC measurements with  $10 \text{ fb}^{-1}$ , the dark matter relic abundance can be calculated to be  $\Omega_{\text{dm}}^{\text{collider}} h^2 = 0.119 \pm 0.033$ . The result is not as precise as the measurement from WMAP, but it is certainly very important whether they are consistent with each other.

*Discussions and conclusion.*—For illustration, we have assumed that the light mediator only decays to muons. The mediator could decay to electrons as well resulting in lepton jets containing electrons. To identify such lepton jets, we need to consider electrons isolated from hadronic activities. Requiring all leptons to be isolated from hadronic jets by  $\Delta R > 0.4$  and taking the electron acceptance into account reduce the efficiency for identifying the signal events to around 34%. The error for the coupling measurement is then increased by a factor of 3. Moreover, if the light mediator only decays to electrons, we will have to determine the mediator’s mass from individual electrons’ momenta in an “electron jet.” For this purpose, relatively soft electrons ( $\leq 10$  GeV) are favored because they can be sufficiently separated by the magnetic field before they hit the electromagnetic calorimeter. Moreover, low momentum electrons are also well measured with the tracker.

In conclusion, we have described how to test a new class of DM models with a long-range dark force at the LHC. The light mediator decays in the detector to a lepton jet, which allows us to measure its mass. The DM mass is determined using the  $m_{T2}$  technique. Moreover, we note

that, when the DM particle is produced, it has a significant rate to radiate an extra light mediator. Therefore, we can extract the coupling constant by measuring the radiation rate. This technique can be generalized to other models with a dark force. For example, the decay chain in the dark sector may be longer than the one we have considered, involving more particles charged under the dark force. In this case, in order to extract the coupling constant, one needs to carefully reconstruct the decay chain and include all contributions to the dark radiation. This and other variations guarantee further studies.

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