Universality of State-Independent Violation of Correlation Inequalities for Noncontextual Theories

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We show that the state-independent violation of inequalities for noncontextual hidden variable theories introduced in [Phys. Rev. Lett. **101**, 210401 (2008)] is universal, i.e., occurs for any quantum mechanical system in which noncontextuality is meaningful. We describe a method to obtain state-independent violations for any system of dimension $d \geq 3$. This universality proves that, according to quantum mechanics, there are no ''classical'' states.

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Introduction.—Bell inequalities [1] are constraints involving the correlations of results of spacelike separated measurements, which are satisfied by any local hidden variable theory, but are violated by entangled states. For years, entanglement has been considered ''*the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought'' [2].

Recently, one of us [3] has shown that, for certain physical systems, there are inequalities for the correlations of compatible measurements which are satisfied by any noncontextual hidden variable theory, but are violated by any quantum state, even by nonentangled and totally mixed states. Specifically, Ref. [3] presents three correlation inequalities, two of which are violated by any state described in quantum mechanics by a Hilbert space of dimension $d = 4$ (i.e., admitting 4 pairwise compatible propositions), and a third which is violated by any state of $d = 2^N$ (with N odd and $N \geq 3$).

An immediate question arises from this result: Can *any* quantum system be shown to violate an inequality which is valid for any noncontextual hidden variable theory? Moreover, does this violation hold for any state? The three inequalities in Ref. [3] are based on three special proofs of the Kochen-Specker (KS) [4,5] theorem in which each observable appears in an even number of contexts, while the prediction of quantum mechanics for the sums or the products of a compatible set of these observables is minus the identity in an odd number of contexts. A related question is therefore: Is there a method to obtain correlation inequalities violated by any quantum state, based on any available proof of the KS theorem, even those without this special property? An affirmative answer would provide a method to obtain state-independent violations for any $d \geq 3$, underlying the universality of this phenomenon. This is the best result possible, since two-dimensional quantum systems can be described by contextual hidden variable theories [4].

The aim of this Letter is to give affirmative answers to these questions. For any quantum system with $d \geq 3$ we find an inequality which is satisfied by observables in any noncontextual hidden variable theory, but violated by their corresponding quantum observables, for any quantum state. We do that by obtaining an inequality from a d-dimensional proof of KS theorem. These results underline the universality of the phenomenon pointed out in [3] and allow us to draw the following conclusions: (i) ''classical'' states are impossible in quantum mechanics, and this impossibility can be tested by experiment; (ii) in this perspective, local realism underlying Bell inequalities can be regarded as noncontextuality restricted to spacelike separated contexts. Thus the inequalities derived here and Bell inequalities belong to a larger family of inequalities satisfied by ''classical'' systems. Bell inequalities are usually optimized to allow for a maximum violation on a particular quantum state, but here we seek inequalities universally violated by all quantum states. The price for this universality is a relatively small maximum degree of violation allowed by quantum mechanics.

Some clarification of the terminology that will be subsequently used might be in order. All the theories which we consider (quantum mechanics, and trivially the noncontextual theories) satisfy the principle of noncontextuality of probability. Suppose that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are physical observables such that $\mathcal{A}$ is compatible with $\mathcal{B}$ and $\mathcal{C}$, but $\mathcal{B}$ is incompatible with $\mathcal{C}$. The principle of noncontextuality of probability states that, for every state, the expectation value of $\mathcal{A}$ is the same whether $\mathcal{A}$ is measured with $\mathcal{B}$, or whether $\mathcal{A}$ is measured with $\mathcal{C}$. That is, the expectation of an observable is context independent. In quantum mechanics, noncontextuality of probability leads to Born's rule (this is Gleason's theorem [6]). Moreover, in quantum mechanics, noncontextuality of probability implies nonsignaling. To see this one can take $\mathcal{A} = \mathbb{1} \otimes \mathcal{A}_2$, $\mathcal{B} = \mathcal{B}_1 \otimes \mathbb{1}$, and $\mathcal{C} = \mathcal{C}_1 \otimes \mathbb{1}$, with $\mathcal{B}_1$ and $\mathcal{C}_1$ incompatible.

Another concept is the noncontextuality of values, which we simply call noncontextuality. With every set of observables in classical theories we can associate numerical values, which are within the range taken by the observables, and respect the algebraic relations among them. In quantum mechanics we cannot do that, and this is the content of the KS theorem [4,5]. Hence, hidden variable theories which associate values with quantum mechanical observables in the above manner must be contextual. The value of an observable is context dependent.

Consider then a physical system admitting d compatible dichotomic observables (values ± 1), hereafter denoted by $A_1, \ldots, A_d$, and consider n different (mutually incompatible) characterizations of this system via observable sets $S^j = \{A_1^j, \ldots, A_d^j\}$ with $j = 1, \ldots, n$. The different S^j's will serve as different contexts. Some of the mutually incompatible sets S^j may overlap, so one can have $j \neq k$, for which $A_i^j = A_m^k$ for some values of i and m. The noncontextuality of probability implies that the expectation of $A_i^j = A_m^k$ remains the same, whether measured within the set (context) S^j or S^k. As noted, the situation is different when one tries to assign noncontextual values to the observables. The KS theorem states that, for $d \geq 3$, there are families of compatible dichotomic observable sets $\{S^j\}_{j=1}^n$, such that it is impossible to consistently assign noncontextual values ± 1 to all the involved observables A_i^j, so that exactly one observable in every set is assigned one of the two values, e.g. -1 and the remaining observables are assigned the other value, e.g. $+1$. Finding a suitable family of sets $\{S^j\}_{j=1}^n$ establishes a proof of the KS theorem.

The question addressed here is whether one can derive from every proof of the KS theorem an inequality for correlations, which will be satisfied by every classical theory (i.e., whenever the A_i^j are interpreted as ± 1 valued classical random variables), and will be violated by the family $\{S^j\}_{j=1}^n$ for *any* quantum state. Such an inequality can be tested in the following way: Fix a quantum state, measure the observables in the set S^1, then prepare another system in the same state and repeat the experiment for S^2, and so on many times. Then, vary the state and repeat it. A detailed description of a complete experiment of this type is presented in [7]. For the procedure to make sense, the inequality must be written as a bound on a function of components, each involving only compatible observables.

Classical inequality.—Consider a physical theory which interprets the observables A_i^j as classical random variables with (simultaneous) noncontextual values ± 1. We shall show that it must satisfy the following inequality:

$$\beta(d, n) \leq n(d-2) - 2, \tag{1}$$

where

$$\beta(d, n) = \sum_{j=1}^{n} \langle B^j \rangle, \tag{2}$$

and

$$B^j = -\sum_{p \neq q} A_p^j A_q^j - \sum_{p \neq q \neq r \neq p} A_p^j A_q^j A_r^j - \cdots - \prod_{k=1}^{d} A_k^j - 1. \tag{3}$$

The proof is as follows:

$$B^j = \sum_{k=1}^{d} A_k^j - \prod_{k=1}^{d} (1 + A_k^j). \tag{4}$$

When $A_k^j = 1$ for all k, then $B^j = d - 2^d$. When, for at least one value of k, $A_k^j = -1$, then $B^j = \sum_{k=1}^d A_k^j$. For any $d > 2$, the former value is smaller than all the latter values. Therefore, $B_{\max}^j = d - 2$, which is obtained for $d - 1$ positive and one negative value of A_k^j. Since the (overlapping) sets S^j are chosen so that it is impossible to produce $B_{\max}^j$ for all j (because $\{S^j\}_{j=1}^n$ yields a proof of the KS theorem), then an upper bound for $\sum_{j=1}^n B^j$ is $n(d-2) - 2$. Therefore, this is also an upper bound for $\sum_{j=1}^n \langle B^j \rangle$.

Quantum violation.—Quantum predictions violate inequality (1). If we associate every A_k^j with a unit vector $|v^{j,k}\rangle$ by

$$A_k^j = \mathbb{1} - 2|v^{j,k}\rangle\langle v^{j,k}|, \tag{5}$$

where $\langle v^{j,k}|v^{j,k'}\rangle = \delta_{kk'}$ for every $1 \leq j \leq n$, then the operator corresponding to the observable B^j is

$$B^j = \sum_{k=1}^{d} A_k^j - \prod_{k=1}^{d} (\mathbb{1} + A_k^j) \tag{6}$$

$$= (d-2)\mathbb{1}, \tag{7}$$

where equality (7) follows from the observation that $\mathbb{1} + A_k^j$ is twice the projection on the $(d-1)$-dimensional subspace orthogonal to $|v^{j,k}\rangle$; hence $\prod_{k=1}^d (\mathbb{1} + A_k^j) = 0$ and thus (7) follows from $\sum_{k=1}^d A_k^j = (d-2)\mathbb{1}$.

By summing Eq. (7) over all the sets S^k, one concludes that, independently of the quantum state, the results of the measurements of the observables B^j lead to a violation of inequality (1). Specifically, according to quantum mechanics,

$$\beta_{\mathrm{QM}} = n(d-2). \tag{8}$$

Impossibility of classical states.—The affirmative answers to our questions show that, for any quantum state, we can design an experiment with an ensemble of particles in this state whose results cannot be reproduced by any noncontextual hidden variable theory. In this sense, all states of physical systems are nonclassical. A totally mixed state is no exception. The measurements performed on totally mixed states, if precise enough, will show their nonclassicality all the same. There is always a finite separation between a classical state and a quantum state. This difference can be observed in actual experiments.

Bell inequalities are a particular case of more general inequalities.—Another consequence of the universality of the state-independent violations of noncontextual inequalities is the following. So far, on one hand, we had Bell inequalities derived from the assumptions of local realism alone, and violated in a state-dependent way by the quantum mechanical predictions. On the other hand, we had proofs of the KS theorem which pointed out a logical contradiction between the assumptions of noncontextual realism and the formal structure of quantum mechanics—and for this very reason, an experimental test of noncontextual realism is a subtle matter indeed. Now, we see that Bell inequalities are, in a sense, the tip of an iceberg. They belong to a more general family of inequalities which are satisfied by appropriate classical random variables, and violated by their corresponding quantum observables. These include inequalities violated not only by entangled states of composite quantum systems, but for any state of any quantum system with $d \geq 3$.

Remarks.—Among all known proofs of the KS theorem for $d = 3$ [4,8–12], the one with the smallest n has $n = 36$ sets and 49 observables [8]. Among all known proofs of the KS theorem for $d = 4$ [12–17], the one with the smallest n has $n = 9$ sets and only 18 observables [15]. There are also proofs for other values of d [18,19], and methods to generate proofs of the KS theorem for any value of d [13,19–21].

For some $\{S^j\}_{j=1}^n$, the upper bound of inequality (1) cannot be reached. For instance, for the 24 observables in $d = 4$ of [12], $n = 24$. The value of β predicted by quantum mechanics is indeed 48 [cf. Eq. (8)], but in this case the upper bound for $\beta(4, 24)$ is 40. It is substantially smaller than the general bound in (1) which is 46. Therefore, the quantum violation is in this case 8, i.e., larger than our universal 2. This is due to the fact that the set of 24 observables in [12] is not critical (i.e., the proof also works if some observables are removed) in the sense that it can generate 96 (critical) 20-observable and 16 (critical) 18-observable proofs of the KS theorem [15].

The case $d = 4$ using the 18 observables in [15] deserves a closer examination. The resulting inequality contains a sum of 99 terms bounded by 16, while the quantum prediction is 18. However, if we omit all the correlations but those between 4 observables, then we obtain a 9-term inequality introduced in [3]. The bound there is 7 while quantum mechanics allows 9. All this shows that the method presented here may lead to inequalities that are not optimal in the sense that they may be strictly weaker than the inequalities with fewer terms but the same violation. Finding simpler inequalities with the same violation is interesting since in actual experiments every expectation value is affected by errors.

Our inequality (1) is related to earlier "KS inequalities" [22,23] between probabilities instead of correlations. Although an equivalence can be established between the final inequalities, the main difference is that, while the derivation of the inequalities in [22,23] assumes the sum rule (i.e., it requires quantum mechanics), the derivation of inequality (1) *only* requires the assumption of noncontextual probabilities (i.e., it does not require quantum mechanics). Quantum mechanics is only used to predict that (1) will be violated by the experimental results.

Conclusions.—To sum it up, we have produced an algorithm associating an inequality for the results of compatible measurements with every proof of the KS theorem. The inequality is satisfied by any noncontextual hidden variable theory. Nevertheless, it is violated by quantum mechanical predictions for every physical state, including the seemingly classical totally mixed state. In this sense, our result shows that there is no such thing as a classical state, and suggests that Bell inequalities are a particular type of a more general inequalities where neither spacelike separation nor entanglement play a fundamental role.

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[1] J. S. Bell, Physics (Long Island City, N.Y.) **1**, 195 (1964).
[2] E. Schrödinger, Proc. Cambridge Philos. Soc. **31**, 555 (1935).
[3] A. Cabello, Phys. Rev. Lett. **101**, 210401 (2008).
[4] S. Kochen and E. P. Specker, J. Math. Mech. **17**, 59 (1967).
[5] N. D. Mermin, Rev. Mod. Phys. **65**, 803 (1993).
[6] A. M. Gleason, J. Math. Mech. **6**, 885 (1957).
[7] E. Amselem, M. Rådmark, M. Bourennane, and A. Cabello (to be published).
[8] K. Schütte in 1965, first reported by K. Svozil in 1994. See also J. Bub, Found. Phys. **26**, 787 (1996).
[9] F. J. Belinfante, *A Survey of Hidden-Variables Theories* (Pergamon Press, New York, 1973). See also E. de Obaldia, A. Shimony, and F. Wittel, Found. Phys. **18**, 1013 (1988).
[10] A. Peres and A. Ron, in *Microphysical Reality and Quantum Formalism*, edited by A. van der Merwe, F. Selleri, and G. Tarozzi (Kluwer, Dordrecht, 1988), Vol. 2, p. 115.
[11] J. H. Conway and S. Kochen around 1990, first reported in A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Dordrecht, 1993), p. 114. See also J. H. Conway and S. Kochen, in *Quantum [Un]speakables:*

From Bell to Quantum Information, edited by R. A. Bertlmann and A. Zeilinger (Springer-Verlag, Berlin, 2002), p. 257.

[12] A. Peres, J. Phys. A **24**, L175 (1991).

[13] J. R. Zimba and R. Penrose, Studies in History and Philosophy of Science **24**, 697 (1993).

[14] M. Kernaghan, J. Phys. A **27**, L829 (1994).

[15] A. Cabello, J. M. Estebaranz, and G. García-Alcaine, Phys. Lett. A **212**, 183 (1996).

[16] R. Penrose, in *Quantum Reflections*, edited by J. Ellis and D. Amati (Cambridge University Press, Cambridge, England, 2000), p. 1.

[17] P. K. Aravind, and F. Lee-Elkin, J. Phys. A **31**, 9829 (1998).

[18] M. Kernaghan and A. Peres, Phys. Lett. A **198**, 1 (1995).

[19] A. Cabello, J. M. Estebaranz, and G. García-Alcaine, Phys. Lett. A **339**, 425 (2005).

[20] A. Cabello and G. García-Alcaine, J. Phys. A **29**, 1025 (1996).

[21] M. Pavičić, J.-P. Merlet, B. D. McKay, and N. D. Megill, J. Phys. A **38**, 1577 (2005).

[22] C. Simon, Č. Brukner, and A. Zeilinger, Phys. Rev. Lett. **86**, 4427 (2001).

[23] J.-Å. Larsson, Europhys. Lett. **58**, 799 (2002).