Symmetries and Supersymmetries of the Dirac Hamiltonian with Axially Deformed Scalar and Vector Potentials

A. Leviatan
Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel
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We consider several classes of symmetries of the Dirac Hamiltonian in 3 + 1 dimensions, with axially deformed scalar and vector potentials. The symmetries include the known pseudospin and spin limits and additional symmetries which occur when the potentials depend on different variables. Supersymmetries are observed within each class and the corresponding charges are identified.

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The Dirac equation plays a key role in microscopic descriptions of many-fermion systems, employing covariant density functional theory and the relativistic mean-field approach. The relevant mean-field potentials are of Coulomb vector type in atoms, and a mixture of Lorentz vector and scalar potentials in nuclei and hadrons [1]. Recently, symmetries of Dirac Hamiltonians with such mixed Lorentz structure have been shown to be relevant for explaining the observed degeneracies of certain shell-model orbitals in nuclei ("pseudospin doublets") [2], and the absence of quark spin-orbit splitting ("spin doublets") [3], as observed in heavy-light quark mesons. Supersymmetric patterns have been identified in specific limits of such spherical potentials [4,5]. In the present Letter we further explore classes of symmetries and supersymmetries when these potentials are axially deformed. Such a study is significant in view of the fact that mean-field Hamiltonians often break the rotational symmetry. Cylindrical geometries are relevant to a number of problems, including electron channeling in crystals, structure of axially deformed nuclei, and quark confinement in spheroidal flux tubes.

The Dirac Hamiltonian, H, for a fermion of mass M moving in external scalar, V_S, and vector, V_V, potentials is given by $H = \hat{\boldsymbol{\alpha}} \cdot \hat{\boldsymbol{p}} + \hat{\beta}(M + V_S) + V_V$ [6]. When the potentials are axially symmetric, i.e., independent of the azimuthal angle ϕ, $V_{S,V} = V_{S,V}(\rho, z)$, $\rho = \sqrt{x^2 + y^2}$, then the z component of the angular momentum operator, $\hat{J}_z$, commutes with H and its half-integer eigenvalues Ω are used to label the Dirac wave functions $\Psi = (g^+ e^{-i\phi/2}, g^- e^{i\phi/2}, if^+ e^{-i\phi/2}, if^- e^{i\phi/2})e^{i\Omega\phi}$. Here $g^\pm \equiv g^\pm(\rho, z)$ and $f^\pm \equiv f^\pm(\rho, z)$ are the radial wave functions of the upper and lower components, respectively. Henceforth, such a wave function will be denoted by $\Psi_\Omega\colon \{g^+, g^-, f^+, f^-\}$. The potentials enter the Dirac equation through the combinations

$$A(\rho, z) = E + M + V_S(\rho, z) - V_V(\rho, z), \tag{1a}$$

$$B(\rho, z) = E - M - V_S(\rho, z) - V_V(\rho, z). \tag{1b}$$

For each solution with $\Omega > 0$, there is a degenerate time-reversed solution with $-\Omega < 0$, hence, we confine the discussion to solutions with $\Omega > 0$. Of particular interest are bound Dirac states with $|E| < M$ and normalizable wave functions in potentials satisfying $\rho V_S(\rho, z)$, $\rho V_V(\rho, z) \to 0$ for $\rho \to 0$ and $V_S(\rho, z)$, $V_V(\rho, z) \to 0$ for $\rho \to \infty$ or $z \to \pm\infty$. The boundary conditions imply that the radial wave functions fall off exponentially for large distances and behave as a power law for $\rho \to 0$. Furthermore, for $z = 0$ and $\rho \to \infty$, $f^-/g^+ \propto (M - E) > 0$ and $g^-/f^+ \propto (M + E) > 0$, while for $z = 0$ and $\rho \to 0$, $f^-/g^+ \propto B(0)\rho$ and $g^-/f^+ \propto -A(0)\rho$. These properties have important implications for the structure of radial nodes. In particular, it follows that for potentials with the indicated asymptotic behavior and $A(0)$, $B(0) > 0$ as encountered in nuclei, a necessary condition for a nodeless bound eigenstate of a Dirac Hamiltonian is

$$g^- = 0 \quad \text{or} \quad f^+ = 0. \tag{2}$$

The Dirac equation, $H\Psi = E\Psi$, leads to a set of four coupled partial differential equations involving the radial wave functions. Their solutions are greatly simplified in the presence of symmetries. We now discuss four classes of relativistic symmetries and possible supersymmetries within each class.

The symmetry of class I, referred to as pseudospin symmetry, occurs when the sum of the scalar and vector potentials is a constant, $V_S(\rho, z) + V_V(\rho, z) = \Delta_0$. The symmetry generators, $\hat{\tilde{S}}_i$, commute with the Dirac Hamiltonian and span an SU(2) algebra [7,8]

$$\hat{\tilde{S}}_i = \begin{pmatrix} U_p \hat{s}_i U_p & 0 \\ 0 & \hat{s}_i \end{pmatrix} \qquad i = x, y, z \qquad U_p = \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{p}. \tag{3}$$

Here $\hat{s}_i = \sigma_i/2$ are the usual spin operators, defined in terms of Pauli matrices. The Dirac eigenfunctions in the pseudospin limit satisfy

$$\hat{\tilde{S}}_z \Psi_\Omega^{(\tilde{\mu})} = \tilde{\mu}\Psi_\Omega^{(\tilde{\mu})}, \qquad \tilde{\mu} = \pm 1/2, \tag{4}$$

and form degenerate SU(2) doublets. Their wave functions have been shown to be of the form [9]

$$\Psi^{(-1/2)}_{\Omega_1 = \tilde{\Lambda} - 1/2}\colon \{g^+, -g, 0, f\}, \tag{5a}$$

$$\Psi^{(1/2)}_{\Omega_2 = \tilde{\Lambda} + 1/2}\colon \{g, g^-, f, 0\}, \tag{5b}$$

where $\tilde{\Lambda} = \Omega - \tilde{\mu} \geq 0$ is the eigenvalue of $\hat{J}_z - \hat{\tilde{S}}_z$. The relativistic pseudospin symmetry has been tested in numerous realistic mean-field calculations of nuclei and were found to be obeyed to a good approximation, especially for doublets near the Fermi surface [9,10]. The dominant upper components of the states in Eq. (5), involving g^+ and g^-, correspond to nonrelativistic pseudospin doublets with asymptotic (Nilsson) quantum numbers $[N, n_3, \Lambda]\Omega = \Lambda + 1/2$ and $[N, n_3, \Lambda + 2]\Omega = \Lambda + 3/2$, respectively. The doublet is expressed in terms of the pseudo-orbital angular momentum projection, $\tilde{\Lambda} = \Lambda + 1$, which is added to the pseudospin projection, $\tilde{\mu} = \pm 1/2$, to form doublet states with $\Omega = \tilde{\Lambda} \pm 1/2$. Such doublets play a crucial role in explaining features of deformed nuclei, including superdeformation and identical bands [9,11].

The symmetry of class II, referred to as spin symmetry, occurs when the difference of the scalar and vector potentials is a constant, $V_S(\rho, z) - V_V(\rho, z) = \Xi_0$. The symmetry group is again SU(2) and its generators [7]

$$\hat{S}_i = \begin{pmatrix} \hat{s}_i & 0 \\ 0 & U_p \hat{s}_i U_p \end{pmatrix}, \qquad i = x, y, z, \tag{6}$$

commute with the Dirac Hamiltonian. The Dirac eigenfunctions in the spin limit satisfy

$$\hat{S}_z \Psi_\Omega^{(\mu)} = \mu \Psi_\Omega^{(\mu)}, \qquad \mu = \pm 1/2, \tag{7}$$

and form degenerate SU(2) doublets. Their wave functions are of the form [9]

$$\Psi^{(1/2)}_{\Omega_1 = \Lambda + 1/2}: \{g, 0, f, f^-\}, \tag{8a}$$

$$\Psi^{(-1/2)}_{\Omega_2 = \Lambda - 1/2}: \{0, g, f^+, -f\}, \tag{8b}$$

where $\Lambda = \Omega - \mu \geq 0$ is the eigenvalue of $\hat{J}_z - \hat{S}_z$. The upper components of the two states in Eq. (8) form the usual nonrelativistic spin doublet with a common radial wave function, g, an orbital angular momentum projection, Λ, and two spin orientations $\Omega = \Lambda \pm 1/2$. The relativistic spin symmetry has been shown to be relevant to the structure of heavy-light quark mesons [3].

The Dirac Hamiltonian has additional symmetries when the scalar and vector potentials depend on different variables. The symmetry of class III occurs when the potentials are of the form $V_S = V_S(z)$ and $V_V = V_V(\rho)$. In this case, the Dirac Hamiltonian commutes with the following Hermitian operator:

$$\hat{R}_z = [M + V_S(z)]\hat{\beta}\hat{\Sigma}_3 + \gamma_5 \hat{p}_z, \tag{9}$$

where $\hat{\Sigma}_i = \left(\begin{smallmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{smallmatrix}\right)$. The Dirac eigenfunctions satisfy

$$\hat{R}_z \Psi_\Omega^{(\epsilon)} = \epsilon \Psi_\Omega^{(\epsilon)}. \tag{10}$$

A separation of variables is possible by choosing the Dirac wave function in the form

$$\Psi_\Omega^{(\epsilon)}: \{u_1 h_+, u_2 h_-, u_1 h_-, -u_2 h_+\}/\sqrt{\rho}, \tag{11}$$

where $u_i \equiv u_i(\rho)$, $h_\pm \equiv h_\pm(z)$ and, for simplicity, we have omitted the label ϵ from these wave functions. The Dirac equation then reduces to a set of two coupled first-order ordinary differential equations in the variable ρ,

$$[d/d\rho - \Omega/\rho]u_1(\rho) - [E - V_V(\rho) + \epsilon]u_2(\rho) = 0, \tag{12a}$$

$$[d/d\rho + \Omega/\rho]u_2(\rho) + [E - V_V(\rho) - \epsilon]u_1(\rho) = 0, \tag{12b}$$

and a separate set in the variable z

$$[M + V_S(z) + d/dz]h_2(z) = \epsilon h_1(z), \tag{13a}$$

$$[M + V_S(z) - d/dz]h_1(z) = \epsilon h_2(z), \tag{13b}$$

where $h_\pm(z) = h_2(z) \pm h_1(z)$. The separation constant, ϵ, plays the role of a mass for the transverse motion and is determined from imposed boundary conditions. A special case within the symmetry class III, with $V_S(z) = 0$ and $\epsilon = \pm\sqrt{M^2 + p_z^2}$, was considered for electron channeling in crystals [12]. For $V_S(z) = 0$, $\hat{R}_z$ of Eq. (9), reduces to the transverse polarization operator relevant to studies of synchrotron radiation in storage rings and QED processes in magnetic flux tubes (e.g., e^+e^- production and Bremsstrahlung) [13].

The symmetry of class IV occurs when the potentials are of the form $V_S = V_S(\rho)$ and $V_V = V_V(z)$. In this case, the following Hermitian operator

$$\hat{R}_\rho = [M + V_S(\rho)]\hat{\Sigma}_3 - i\hat{\beta}\gamma_5(\hat{\boldsymbol{\Sigma}} \times \hat{\boldsymbol{p}})_3 \tag{14}$$

commutes with the Dirac Hamiltonian and the Dirac eigenfunctions satisfy

$$\hat{R}_\rho \Psi_\Omega^{(\tilde{\epsilon})} = \tilde{\epsilon} \Psi_\Omega^{(\tilde{\epsilon})}. \tag{15}$$

Again, a separation of variables is possible with the choice of wave function,

$$\Psi_\Omega^{(\tilde{\epsilon})}: \{\xi_1 w_+, -i\xi_2 w_-, i\xi_1 w_-, -\xi_2 w_+\}/\sqrt{\rho}, \tag{16}$$

where $\xi_i \equiv \xi_i(\rho)$ and $w_\pm \equiv w_\pm(z)$. The Dirac equation then reduces to a set of ordinary differential equations in the variable ρ,

$$[d/d\rho - \Omega/\rho]\xi_1(\rho) - [\tilde{\epsilon} + M + V_S(\rho)]\xi_2(\rho) = 0, \tag{17a}$$

$$[d/d\rho + \Omega/\rho]\xi_2(\rho) + [\tilde{\epsilon} - M - V_S(\rho)]\xi_1(\rho) = 0, \tag{17b}$$

and a separate set in the variable z

$$[E - V_V(z) - id/dz]w_2(z) = \tilde{\epsilon} w_1(z), \tag{18a}$$

$$[E - V_V(z) + id/dz]w_1(z) = \tilde{\epsilon} w_2(z), \tag{18b}$$

where $w_\pm(z) = w_2(z) \pm w_1(z)$. The quantum number, $\tilde{\epsilon}$, plays the role of an energy for the transverse motion. A particular selection of potentials within the symmetry class IV was encountered in the study of the Schwinger mechanism for particle-production in a strong confined field $[V_V(z) = \alpha_V z]$ [14,15], $q\bar{q}$ pair creation in a flux tube $[V_S(\rho) = 0, \tilde{\epsilon} = \pm\sqrt{M^2 + k^2}]$ [16], and the canonical quantization in cylindrical geometry of a free Dirac field $[V_S(\rho) = V_V(z) = 0, E^2 = M^2 + k^2 + p_z^2]$ [17].

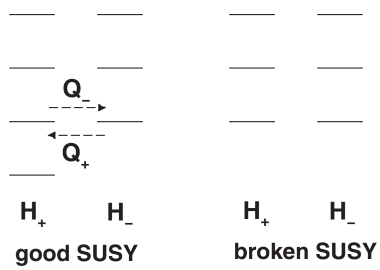

FIG. 1. Typical spectra of good and broken SUSY. The operators Q_- and Q_+ connect degenerate states in the H_+ and H_- sectors.

Dirac Hamiltonians with selected external fields are known to be supersymmetric [4–6,18]. It is, therefore, natural to inquire whether a supersymmetric structure can develop within each of the above symmetry classes. The essential ingredients of supersymmetric quantum mechanics [18] are the supersymmetric Hamiltonian, $\mathcal{H}$, and charges Q_+, $Q_- = Q_+^\dagger$, which generate the supersymmetry (SUSY) algebra $[\mathcal{H}, Q_\pm] = \{Q_\pm, Q_\pm\} = 0$, $\{Q_-, Q_+\} = \mathcal{H}$. Accompanying this set is a Hermitian Z_2-grading operator satisfying $[\mathcal{H}, \mathcal{P}] = \{Q_\pm, \mathcal{P}\} = 0$ and $\mathcal{P}^2 = \mathbb{1}$. The $+1$ and -1 eigenspaces of $\mathcal{P}$ define the "positive-parity," H_+, and "negative parity," H_-, sectors of the spectrum, with eigenvectors $\Psi^{(+)}$ and $\Psi^{(-)}$, respectively. The SUSY algebra imply that if $\Psi^{(+)}$ is an eigenstate of $\mathcal{H}$, then also $\Psi^{(-)} = Q_-\Psi^{(+)}$ is an eigenstate of $\mathcal{H}$ with the same energy, unless $Q_-\Psi^{(+)}$ vanishes or produces an unphysical state (e.g., non-normalizable). The resulting spectrum consists of pairwise degenerate levels with a nondegenerate single state (the ground state) in one sector when the supersymmetry is exact. If all states are pairwise degenerate, the supersymmetry is said to be broken. Typical spectra for good and broken SUSY are shown in Fig. 1. Degenerate doublets, signaling a supersymmetric structure, can emerge in a quantum system with a Hamiltonian H, from the existence of two Hermitian, conserved and anticommuting operators, $\hat{R}$ and $\hat{B}$

$$[H, \hat{R}] = [H, \hat{B}] = \{\hat{R}, \hat{B}\} = 0. \tag{19}$$

The operator $\hat{R}$ has nonzero eigenvalues, r, which come in pairs of opposite signs. $\hat{B}^2 = \hat{B}^\dagger \hat{B} = f(H)$, is a function of the Hamiltonian. A Z_2-grading operator, $\mathcal{P}_r = \hat{R}/|r|$, and Hermitian supercharges $Q_1 = \hat{B}$, $Q_2 = iQ_1\mathcal{P}_r$ can now be constructed. The triad of operators $Q_\pm = (Q_1 \pm iQ_2)/2$ and $\mathcal{H} = Q_1^2 = f(H)$ form the standard SUSY algebra. In the present analysis, $f(H)$ is a quadratic function of the Dirac Hamiltonian, H, and the relevant $\hat{R}$ and $\hat{B}$ operators are listed in Table I.

In the pseudospin symmetry limit, the relevant operator $\hat{B}$, connects the doublet states of Eq. (5). The spectrum, for each $\tilde{\Lambda} \neq 0$, consists of twin towers of pairwise degenerate pseudospin doublet states, with $\Omega_1 = \tilde{\Lambda} - 1/2$ and $\Omega_2 = \tilde{\Lambda} + 1/2$, and an additional nondegenerate nodeless state at the bottom of the $\Omega_1 = \tilde{\Lambda} - 1/2$ tower. Such nodeless states correspond in the nonrelativistic nuclear deformed shell-model to the "intruder" states, $[N, n_3, \Lambda = N - n_3]\Omega = \Lambda + 1/2$, which, empirically, are found not to be part of a doublet [11]. The latter property follows from the fact that a nodeless bound Dirac state satisfies the criteria of Eq. (2), hence has a wave function as in Eq. (5a) with g^+, g, $f \neq 0$, and $f/g^+ > 0$. Its pseudospin partner state has a wave function as in Eq. (5b). The radial components satisfy $Bg^- = [B - 2(\tilde{\Lambda}/\rho)f/g^+]g^+$, where B is defined in Eq. (1b). This relation is satisfied, to a good approximation, for mean-field potentials relevant to nuclei, and the right-hand side (rhs) is nonzero and, consequently, $g^- \neq 0$. If so, then the partner state (5b) is also nodeless, but it cannot be a bound eigenstate since its radial components do not fulfill the condition of Eq. (2). Altogether, the ensemble of Dirac states with $\Omega_2 - \Omega_1 = 1$ exhibits a supersymmetric pattern of good SUSY, as illustrated in Fig. 2(a).

In the spin symmetry limit, the relevant operator $\hat{B}$ connects the doublet states of Eq. (8). The spectrum, for each $\Lambda \neq 0$, consists of twin towers of pairwise degenerate spin-doublet states with $\Omega_1 = \Lambda - 1/2$ and $\Omega_2 = \Lambda + 1/2$. None of these towers have a single nondegenerate state. This follows from the fact that, in view of Eq. (2), a nodeless bound state has a wave function as in Eq. (8a) with g, f, $f^- \neq 0$ and $g/f^- > 0$. Its spin partner has a wave function as in Eq. (8b). The radial components satisfy $Af^+ = [A - 2(\Lambda/\rho)g/f^-]f^-$, where A is defined in Eq. (1a). For relevant potentials the rhs of this relation can vanish, hence f^+ has a node. Therefore, the spin partner of a nodeless state is not nodeless and can be a bound eigenstate, since the restrictions of Eq. (2) do not apply. Altogether, the ensemble of Dirac states with $\Omega_2 - \Omega_1 = -1$ exhibits a supersymmetric pattern of broken SUSY, as illustrated in Fig. 2(b).

Within the symmetry class III, a supersymmetry is obtained for $V_V(\rho) = \alpha_V/\rho$ and $V_S(z)$ arbitrary. The energy

TABLE I. Conserved, anticommuting operators for Dirac Hamiltonians (H) exhibiting a supersymmetric structure.

SUSY	$\hat{R}$	$\hat{B}$	$\hat{B}^2 = f(H)$
$V_S(\rho, z) + V_V(\rho, z) = \Delta_0$	$\hat{\tilde{S}}_z$ (3)	$2(M + \Delta_0 - H)\hat{\tilde{S}}_x$	$(M + \Delta_0 - H)^2$
$V_S(\rho, z) - V_V(\rho, z) = \Xi_0$	$\hat{S}_z$ (6)	$2(M + \Xi_0 + H)\hat{S}_x$	$(M + \Xi_0 + H)^2$
$V_S = V_S(z)$, $V_V = \frac{\alpha_V}{\rho}$	$\hat{R}_z$ (9)	$\hat{\beta}\hat{\Sigma}_3\{i\hat{J}_z\gamma_5[H - \hat{\Sigma}_3\hat{R}_z] - \frac{\alpha_V}{\rho}(\hat{\mathbf{\Sigma}} \cdot \boldsymbol{\rho})\hat{R}_z\}$	$\hat{J}_z^2(H^2 - \hat{R}_z^2) + \alpha_V^2\hat{R}_z^2$
$V_S = \frac{\alpha_S}{\rho}$, $V_V = V_V(z)$	$\hat{R}_\rho$ (14)	$\hat{\Sigma}_3\{i\hat{J}_z\gamma_5[M - \hat{\Sigma}_3\hat{R}_\rho] - \frac{\alpha_S}{\rho}(\hat{\mathbf{\Sigma}} \cdot \boldsymbol{\rho})\hat{\beta}\hat{R}_\rho\}$	$\hat{J}_z^2(\hat{R}_\rho^2 - M^2) + \alpha_S^2\hat{R}_\rho^2$

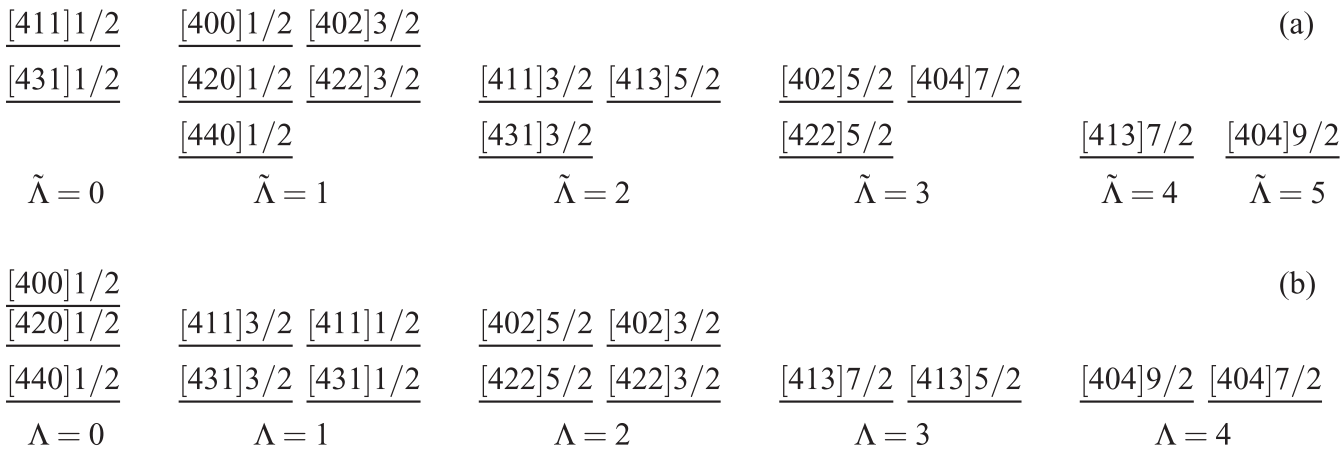

FIG. 2. Grouping of deformed shell-model states $[N = 4, n_3, \Lambda]\Omega$, exhibiting a pattern of (a) good SUSY, relevant to the pseudospin symmetry limit, and (b) broken SUSY, relevant to the spin symmetry limit. N and n_3 are harmonic oscillator quantum numbers. Λ ($\tilde{\Lambda}$) is the orbital (pseudo-orbital) angular momentum projection along the symmetry z axis.

eigenvalues are $E^{(\epsilon)}_{n_\rho,\Omega} = |\epsilon|/\sqrt{1 + \alpha_V^2/(n_\rho + \gamma)^2}$ ($n_\rho = 0, 1, 2, \ldots$), with $\gamma = \sqrt{\Omega^2 - \alpha_V^2}$. From Eqs. (13) we see that if $[h_1(z), h_2(z)]$ are solutions with $\epsilon > 0$, then $[h_1(z), -h_2(z)]$ are solutions with $-\epsilon < 0$. Accordingly, the doublet wave functions are as in Eq. (11), with the replacements, $u_i \mapsto u_i^{(\epsilon)}(\rho)$ for $\Psi^{(\epsilon)}_{n_\rho,\Omega}$, and $u_i \mapsto u_i^{(-\epsilon)}(\rho)$, $h_\pm \mapsto -h_\mp(z)$ for $\Psi^{(-\epsilon)}_{n_\rho,\Omega}$. For $n_\rho \geq 1$, the states $\Psi^{(\pm\epsilon)}_{n_\rho,\Omega}$ are degenerate. For $n_\rho = 0$ only one state is an acceptable solution, which has $\epsilon > 0$ (assuming $\alpha_V < 0$) and is annihilated by the relevant operator $\hat{B}$. For each Ω and ϵ the spectrum resembles a supersymmetric pattern of good SUSY, with the towers H_+ (H_-) of Fig. 1 corresponding to states with $\epsilon > 0$ ($\epsilon < 0$).

Within the symmetry class IV, a supersymmetry is obtained for $V_S(\rho) = \alpha_S/\rho$ ($\alpha_S < 0$) and $V_V(z)$ arbitrary. The allowed values are $\tilde{\epsilon} = \pm M\sqrt{1 - \alpha_S^2/(n_\rho + \tilde{\gamma})^2}$ ($n_\rho = 0, 1, 2, \ldots$), where $\tilde{\gamma} = \sqrt{\Omega^2 + \alpha_S^2}$. From Eqs. (18) we see that if $[w_1(z), w_2(z)]$ are solutions with $\tilde{\epsilon} > 0$, then $[w_1(z), -w_2(z)]$ are solutions with $-\tilde{\epsilon} < 0$ and the same energy, E. Accordingly, the doublet wave functions are as in Eq. (16), with the replacements, $\xi_i \mapsto \xi_i^{(\tilde{\epsilon})}(\rho)$ for $\Psi^{(\tilde{\epsilon})}_{n_\rho,\Omega}$, and $\xi_i \mapsto \xi_i^{(-\tilde{\epsilon})}(\rho)$, $w_\pm \mapsto -w_\mp(z)$ for $\Psi^{(-\tilde{\epsilon})}_{n_\rho,\Omega}$. For $n_\rho \geq 1$ the states $\Psi^{(\pm\tilde{\epsilon})}_{n_\rho,\Omega}$ are degenerate. For $n_\rho = 0$ only one state, with $\tilde{\epsilon} > 0$, is an acceptable solution, which is annihilated by the relevant operator $\hat{B}$. Again, for each Ω and $\tilde{\epsilon}$ the resulting spectrum resembles a supersymmetric pattern of good SUSY.

In summary, we have considered classes of symmetries and related supersymmetries of Dirac Hamiltonians with cylindrically deformed scalar and vector potentials. The symmetries arise when the potentials obey a constraint on their sum or difference, or when they depend on different variables. The known pseudospin and spin symmetry limits are by themselves supersymmetric. Additional supersymmetries arise when one of the potentials has a $1/\rho$ dependence and the second potential depends on z. It is gratifying to note that some of the indicated (super)symmetries are manifested empirically, to a good approximation, in physical dynamical systems.

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