Universal Generation of Statistical Self-Similarity: A Randomized Central Limit Theorem

Iddo Eliazar[1,*] and Joseph Klafter[2,3,†]

[1]Department of Technology Management, Holon Institute of Technology, P.O. Box 305, Holon 58102, Israel
[2]School of Chemistry, Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel
[3]Freiburg Institute for Advanced Studies (FRIAS), University of Freiburg, 79104, Freiburg, Germany

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A universal mechanism for the generation of statistical self-similarity—i.e., fractality in the context of random processes—is established. We consider a generic system which superimposes independent stochastic signals, producing a system output; all signals share a common statistical signal pattern, yet each signal has its own transmission parameters—amplitude, frequency, and initiation epoch. We characterize the class of parameter randomizations yielding statistically self-similar outputs in a universal fashion—i.e., for whatever signals fed into the system. Statistically self-similar outputs with finite variance further display (i) anomalous diffusion behavior—characterized by power-law temporal variance growth—and (ii) $1/f$ noise behavior—characterized by power-law power spectra. The mechanism presented is a "randomized central limit theorem" for fractal statistics of random processes.

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Fractal objects and structures—characterized by geometries which are invariant under changes of scale—are ubiquitous across the sciences and play a central role in physics [1]. In the context of stochastic dynamics, fractality is defined via the notion of statistical self-similarity [2]: A random process $\xi = [\xi(t)]_{t\geq 0}$ is said to be statistically self-similar with Hurst exponent H if, for any positive scale c, the random processes $[\xi(ct)]_{t\geq 0}$ and $[c^H \xi(t)]_{t\geq 0}$ are equal in law. This means that speeding up the random process ξ by the factor c is statistically equivalent to scaling it up by the factor c^H. Visually, zooming in and out on the sample-path trajectory of a statistically self-similar random process yields, statistically, the same "picture"—parts of the trajectory, once scaled, look like the whole trajectory.

Four key examples of statistically self-similar random processes are (i) Brownian motion—the random and erratic motion of diffusion—apparently the most fundamental and abundant type of stochastic dynamics encountered in nature; (ii) fractional Brownian motions [3]; (iii) Lévy motions [4]; (iv) fractional stable Lévy motions [5,6].

In this Letter, we study the origins of statistical self-similarity and establish a universal mechanism for the generation of this phenomenon. In what follows, we construct a stochastic superposition model—a system receiving random input signals and producing a random output process—which generates statistical self-similarity in a universal "central limit theorem fashion": The system yields statistically self-similar output processes for whatever input signals fed into it.

Statistical self-similarity is prevalent in Internet-age information traffic [7,8]—especially in the case of major communication channels and routers fed by numerous transmission sources. This empirical observation motivates the modeling of systems which superimpose many signal processes—aggregating them up to produce a collective system-output process. In this research we consider a generic such system which superimposes a multitude of independent signal processes: all signal processes governed by a common—yet arbitrary—stochastic dynamics and thus sharing a common statistical signal pattern; each signal process having its own transmission parameters—amplitude, frequency, and initiation epoch.

Specifically, process k transmits the signal pattern $X_k = [X_k(t)]_{t\geq 0}$ with amplitude a_k (real valued), frequency ω_k (positive valued), and initiation epoch τ_k (non-negative valued). The system superimposes all signal processes—yielding the output process $Y = [Y(t)]_{t\geq 0}$ given by

$$Y(t) = \sum_{\tau_k \leq t} a_k X_k(\omega_k(t - \tau_k)). \tag{1}$$

The processes' signal patterns $\{X_k\}_k$ are assumed independent and identically distributed (i.i.d.) copies of a common generic signal pattern $X = [X(t)]_{t\geq 0}$. The superposition model of Eq. (1) can be regarded as a stochastic transformation mapping the random input signal pattern X to the random output process Y.

The origins of the superposition model of Eq. (1) can be traced back to the comprehensive studies of Rice on shot noise [9,10]: Setting $X(t) = \varphi(t)$—where $\varphi(t)$ is an impulse-response function decaying to zero $[\lim_{t\to\infty} \varphi(t) = 0]$—renders the output Y a shot noise process. The superposition model of Eq. (1) is also related to wavelet analysis [11]: Setting $X(t) = \varphi(t)$—where $\varphi(t)$ is a mother wavelet—renders the output process Y a random wavelet expansion. Yet another example of a deterministic signal pattern is $X(t) = \sin(t)$—rendering the output process Y a random superposition of harmonic signals.

The processes' transmission parameters $\mathcal{P} = \{(a_k, \omega_k, \tau_k)\}_k$ form a collection of points scattered arbitrarily on the three-dimensional domain $\mathcal{D} = (-\infty, \infty) \times (0, \infty) \times [0, \infty)$. In large systems—as considered here—it is natural to assume that the parameters $\mathcal{P}$ follow some statistical regularity. The common statistical method for the random scattering of points in general domains is that of Poisson point processes [12]. Poisson point processes have a wide spectrum of applications ranging from insurance and finance [13] to queueing systems [14]. In recent years, we applied Poissonian randomizations in various topics in statistical physics, obtaining results which are unattainable by i.i.d. randomizations—see [15,16], and references therein.

Henceforth, the parameters $\mathcal{P}$ are assumed a Poisson point process with intensity $\Lambda(x, y, z)$. Informally, this means that a signal process with transmission parameters (a, ω, τ) belonging to the infinitesimal box $(x, x + dx) \times (y, y + dy) \times (z, z + dz)$ exists with probability $\Lambda(x, y, z)dxdydz$. More precisely, this means that [12]: (i) the number of signal processes with transmission parameters residing in a subdomain D (of the domain $\mathcal{D}$) is Poisson-distributed with mean $\iiint_D \Lambda(x, y, z)dxdydz$; (ii) the number of signal processes with transmission parameters residing in disjoint subdomains (of the domain $\mathcal{D}$) are independent random variables. The Poissonian intensity $\Lambda(x, y, z)$ governs the statistics of the transmission parameters $\mathcal{P}$.

The theoretical question we address in this Letter is the following: Can the transmission parameters $\mathcal{P}$ be randomized so that the output process Y will always be statistically self-similar—regardless of the choice of the input signal pattern X? The answer is affirmative. Analysis shows that the output process Y is statistically self-similar with Hurst exponent H—regardless of the choice of the input signal pattern X—if and only if the Poissonian intensity $\Lambda(x, y, z)$ satisfies the scaling relation

$$\Lambda\left(c^H x, \frac{y}{c}, cz\right) = \frac{1}{c^H}\Lambda(x, y, z) \tag{2}$$

[$(x, y, z) \in \mathcal{D}$; c being an arbitrary positive scale]. For example, Poissonian intensities of the form $\Lambda(x, y, z) = x^{\varepsilon_1} y^{\varepsilon_2} z^{\varepsilon_3}$ satisfy Eq. (2) with $H = (\varepsilon_2 - \varepsilon_3)/(1 + \varepsilon_1)$.

Equation (2) establishes a universal mechanism for the generation of statistical self-similarity: No matter what input signal pattern X is fed into the system, a proper Poissonian randomization of the transmission parameters will always yield an output process Y which is statistically self-similar with Hurst exponent H. If we regard the input signal pattern X and the output process Y, respectively, as the system's microscopic and macroscopic levels, then universality is attained by transcending from the system's microscopic level to its macroscopic level. Equation (2) also establishes yet another connection between Poisson point processes and stochastic fractality [17–19].

The Poissonian randomization method is reminiscent of the central limit theorem (CLT): By using a proper deterministic scaling, the aggregates of i.i.d. random variables (with finite variance) always converge to the universal Gaussian law—regardless of the choice of the random variables' probability law. Here the i.i.d. signal patterns $\{X_k\}_k$ replace the CLT i.i.d. random variables, the superposition of Eq. (1) replaces the CLT aggregate, and the random transmission parameters $\mathcal{P}$ replace the CLT deterministic scaling. Because of this analogy, the Poissonian randomization method can be regarded as a "randomized central limit theorem" for statistical self-similarity.

It should be emphasized, however, that the superposition of Eq. (1) does not necessarily span the entire class of statistically self-similar random processes. The mechanism established here is "universal" in the sense that all input signal patterns X (when properly randomized) yield statistically self-similar output processes Y—rather than in the sense that it is capable of generating all statistically self-similar processes. Indeed, to generate a specific statistically self-similar random process (e.g., a fractional Brownian motion), all of its multidimensional marginal distributions need to be met—rather than only its self-similarity structure (which is characterized by a one-dimensional parameter—the Hurst exponent H). To that end, the mechanism presented here should be distinguished from models yielding stochastic convergence to fractional Brownian motions—see [20,21], and references therein.

Let us examine now the case of self-similar output processes with finite variance. In this case, scaling arguments imply that if the output process Y is statistically self-similar with Hurst exponent H, then (i) its temporal variance growth follows the power law

$$\langle Y(t)^2\rangle - \langle Y(t)\rangle^2 = C_V t^{2H} \tag{3}$$

($t \geq 0$; C_V being a positive constant); (ii) its power spectrum follows the power law

$$\lim_{T\to\infty} \frac{1}{T}\left\langle \left| \int_0^T \exp(ift)Y(t)dt \right|^2 \right\rangle = C_S|f|^{-2H-1} \tag{4}$$

(f real; C_S being a positive constant).

The temporal power-law variance growth of Eq. (3) characterizes anomalous diffusion [22,23]—subdiffusive in the Hurst range $H < \frac{1}{2}$ and superdiffusive in the Hurst range $H > \frac{1}{2}$. Also, the power-law power spectrum of Eq. (4) characterizes $1/f$ noise ("flicker" noise) [24,25]. Both anomalous diffusion and $1/f$ noise are statistical phenomena ubiquitously observed across various fields of science and engineering and are the hallmarks of nondiffusive transport [26]. Equation (2) is thus—in the case of finite variance—also a universal mechanism for the generation of these statistical phenomena.

By using probabilistic conditioning and results from the theory of Poisson processes [Ref. [12], Eqs. (3.9)–(3.10)], it can be shown that a necessary and sufficient condition for the self-similar output process Y to possess a finite variance is

$$\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^1 [x^2\psi_X(y(1-z))]\Lambda(x, y, z)dxdydz < \infty, \tag{5}$$

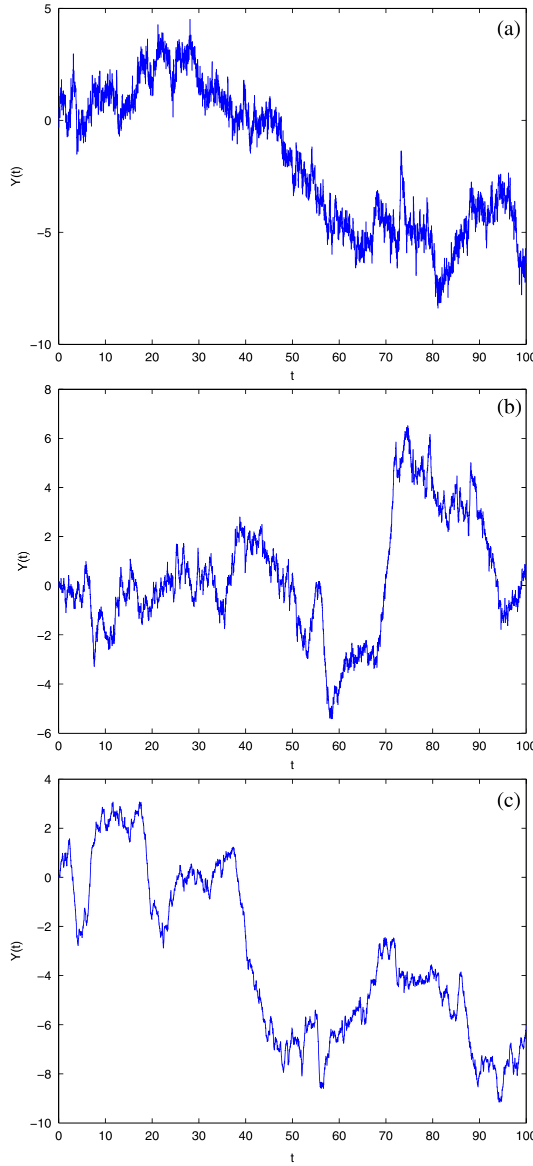

FIG. 1 (color online). Monte Carlo simulations of sample-path trajectories of the output process Y. Poissonian randomization: The intensity of Eq. (8) with $\phi(s) = \exp(-s)$ and $\varepsilon = 0$—the shot noise scenario. Number of superimposed signal processes: Poisson-distributed with mean $\mu = 50\,000$. Input signal pattern: $X(t) = \exp(-t)\sin(t)$. Hurst exponents: $H = 0.3$ in (a), $H = 0.5$ in (b), and $H = 0.7$ in (c).

where $\psi_X(t) = \langle X(t)^2 \rangle$ $(t \geq 0)$ is the mean square displacement of the signal pattern X. We note that the finite-variance condition of Eq. (5) does depend on the choice of the input signal pattern X.

Consider now Poissonian intensities which decouple time and amplitude frequency—i.e., intensities admitting the functional form $\Lambda(x, y, z) = \lambda(x, y)\eta(z)$. For the scaling relation of Eq. (2) to hold, $\eta(z)$ is required to be a homogeneous function. Also, if the function $\eta(z)$ is homogeneous with exponent ε—i.e., $\eta(cz) = c^{\varepsilon}\eta(z)$ (z nonnegative; c positive; ε real)—then (i) the scaling relation of Eq. (2) holds if and only if the function $\lambda(x, y)$ satisfies the scaling relation

$$\lambda\left(c^H x, \frac{y}{c}\right) = \frac{1}{c^{H+\varepsilon}}\lambda(x, y) \tag{6}$$

(x real; y and c positive); (ii) the finite-variance condition of Eq. (5) is met if and only if

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}[x^2\Psi_X(y; \varepsilon)]\lambda(x, y)dxdy < \infty, \tag{7}$$

where $\Psi_X(y; \varepsilon) = \int_0^1 \psi_X(yu)(1-u)^{\varepsilon}du$ (y positive).

A special case of decoupled Poissonian intensities is the "shot noise scenario" in which signal processes with amplitude a and frequency ω appear, randomly in time, at rate $\lambda(a, \omega)$; in this special case $\eta(z) \equiv 1$, $\varepsilon = 0$, and the finite-variance condition of Eq. (7) holds with $\Psi_X(y; 0) = \int_0^1 \psi_X(yu)du$. Yet another special case of decoupled Poissonian intensities is the "simultaneous initiation scenario" in which all signal processes initiate at time zero (i.e., $\tau_k = 0$ for all k); in this special case $\eta(z)$ is Dirac's delta function, $\varepsilon = -1$, and the finite-variance condition of Eq. (7) holds with the function $\psi_X(y)$ replacing the function $\Psi_X(y; \varepsilon)$.

An entire class of functions $\lambda(x, y)$ satisfying the scaling relation of Eq. (6) is given by

$$\lambda(x, y) = \phi(|x|y^H)y^{H+\varepsilon}, \tag{8}$$

where $\phi(s)$ $(s \geq 0)$ is a positive valued function. The intensities of Eq. (8) meet the finite-variance condition of Eq. (7) if and only if the following three conditions are satisfied: (C1) Exponents: $-1 \leq \varepsilon < 2H$, where the exponent value $\varepsilon = -1$ corresponds to the simultaneous initiation scenario. (C2) Poissonian randomization: $\int_0^\infty \phi(s)s^2 ds < \infty$. (C3) Input mean square displacement: $\int_0^\infty \psi_X(t)t^{\varepsilon-2H}dt < \infty$.

Since condition (C3) is input-dependent, each input signal pattern X has a different range of Hurst exponents

TABLE I. The class of Poissonian intensities of Eq. (8): examples of deterministic input signal patterns X and their corresponding finite-variance Hurst ranges. The parameters p and q appearing in examples 3–5 are, respectively, positive and real; in examples 3 and 5, it is required that $q > \varepsilon$.

	Input signal $X(t) =$	Hurst range
1	$\sin(t)$	$\frac{1+\varepsilon}{2} < H < \frac{3+\varepsilon}{2}$
2	$\exp(-t)\sin(t)$	$\max\{0, \frac{\varepsilon}{2}\} < H < \frac{3+\varepsilon}{2}$
3	$\exp(-t^p)t^{q-1}$	$\max\{0, \frac{\varepsilon}{2}\} < H < \frac{q+\varepsilon}{2}$
4	$\exp(-t^{-p})t^{q-1}$	$\max\{0, \frac{\varepsilon}{2}, \frac{q+\varepsilon}{2}\} < H < \infty$
5	$(1+t)^{-p}t^{q-1}$	$\max\{0, \frac{\varepsilon}{2}, \frac{q-p+\varepsilon}{2}\} < H < \frac{q+\varepsilon}{2}$

for which the self-similar output process Y is of finite variance. Finite-variance Hurst ranges for various examples of deterministic input signal patterns X are given in Table I. Monte Carlo simulations of sample-path trajectories of the output process Y, for Example 2 of Table I [the deterministic signal pattern $X(t) = \exp(-t)\sin(t)$, with $\phi(s) = \exp(-s)$ and $\varepsilon = 0$—the shot noise scenario], are depicted in Fig. 1: $H = 0.3$ in Fig. 1(a), $H = 0.5$ in Fig. 1(b), and $H = 0.7$ in Fig. 1(c).

As an example of a stochastic signal pattern X, consider the Ornstein-Uhlenbeck process—the stochastic motion governed by the Langevin equation [27]. The mean square displacement of the Ornstein-Uhlenbeck process is of the form $\psi_X(t) = \upsilon[1 - \exp(-\kappa t)]$ (υ and κ being positive parameters), and hence the corresponding Hurst range is $\frac{1+\varepsilon}{2} < H < \frac{2+\varepsilon}{2}$. In the shot noise scenario $\varepsilon = 0$, and the output process Y is thus superdiffusive with $\frac{1}{2} < H < 1$. Also, in the simultaneous initiation scenario $\varepsilon = -1$, and the output process Y is thus subdiffusive with $0 < H < \frac{1}{2}$.

This Letter established a universal mechanism for the generation of fractality in the context of random processes. The mechanism is a randomized central limit theorem for fractal statistics of random processes: self-similarity, anomalous diffusion, and $1/f$ noise. For the detailed proofs of the results presented in this Letter, readers are referred to Ref. [28].

We considered the generic system model of Eq. (1): the superposition of many independent stochastic signal processes, all processes sharing a common statistical signal pattern X, yet each process having its own transmission parameters—amplitude, frequency, and initiation epoch. Considering randomized transmission parameters, we proved that the system's superimposed output process Y is statistically self-similar—for whatever signal processes fed into the system—if and only if the parameter randomization is governed by Poissonian intensities satisfying the scaling relation of Eq. (2).

In the case of finite-variance output processes, statistical self-similarity further induces both anomalous diffusion and $1/f$ noise behaviors—the hallmarks of nondiffusive transport. The statistically self-similar output processes were shown to be of finite variance if and only if the integrability condition of Eq. (5) is met—in which case the anomalous diffusion and $1/f$ noise behaviors are given, respectively, by Eqs. (3) and (4). Last, special cases of Poissonian intensities which decouple time and amplitude frequency were analyzed and exemplified.

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*eliazar@post.tau.ac.il
†klafter@post.tau.ac.il

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[28] See EPAPS Document No. E-PRLTAO-103-056931 for proofs of the main results presented in this Letter. For more information on EPAPS, see http://www.aip.org/pubservs/epaps.html.