Persistent Current in Small Superconducting Rings

Georg Schwiete[1] and Yuval Oreg[1,2]

[1]Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot, 76100, Israel
[2]Department of Applied Physics, Stanford University, Stanford, California 94305, USA

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We study theoretically the contribution of fluctuating Cooper pairs to the persistent current in superconducting rings threaded by a magnetic flux. For sufficiently small rings, in which the coherence length ξ exceeds the radius R, mean field theory predicts a full reduction of the transition temperature to zero near half-integer flux. We find that nevertheless a very large current is expected to persist in the ring as a consequence of Cooper pair fluctuations that do not condense. For larger rings with $R \gg \xi$, we calculate analytically the susceptibility in the critical region of strong fluctuations and show that it reflects competition of two interacting complex order parameters.

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Introduction and main results.—Superconducting fluctuations have been the subject of intense research during the past decades [1]. At temperatures above the transition temperature T_c to the superconducting state, when the system is still metallic, pairs of electrons are formed for a limited time. These superconducting fluctuations affect both transport and thermodynamic properties.

In bulk superconductors, T_c can be reduced or even completely suppressed by various phase-breaking mechanisms, for example, by applying a magnetic field or introducing magnetic impurities. A special situation occurs for superconducting rings and cylinders threaded by a magnetic flux ϕ. T_c is *periodically* reduced as a function of ϕ, a phenomenon known as Little-Parks oscillations [2]. The period of the oscillations is equal to 1 as a function of the reduced flux $\varphi = \phi/\phi_0$, where the superconducting flux quantum is $\phi_0 = \pi/e$ [3]; see Fig. 1.

The magnitude of the maximal reduction in T_c is size-dependent. As we see in Fig. 1, mean field (MF) theory predicts that for small rings or cylinders with $r \equiv R/\xi < 0.6$ the transition temperature is equal to zero in a finite interval close to half-integer flux, giving rise to a flux-tuned quantum phase transition; see also Eq. (8) below. In this Letter, we show that the pair fluctuations give a large contribution to the persistent current (PC) I even at fluxes for which T_c is reduced to zero and the system has a finite resistance.

Recent experiments added significantly to our understanding of fluctuation phenomena in superconductors with doubly connected geometry. Strong Little-Parks oscillations in the region where $\xi > R$, where T_c is reduced to zero, have been observed in a transport measurement on superconducting cylinders [4]. Koshnick *et al.* [5] measured the PC in small superconducting rings in the regime where $R > \xi$; for the smallest rings under study, T_c was reduced by $\approx 6\%$.

In this Letter, we discuss the PC in the regime of both moderate T_c suppression for $r = R/\xi \gtrsim 1$ as well as the strong Little-Parks oscillations for $r < 0.6$. Before presenting the details of our approach, we summarize the main results of our analysis.

I. Regime with $r = R/\xi < 0.6$.—For $r < 0.6$, the mean field T_c vanishes in a finite interval of fluxes $(\varphi_c, 1-\varphi_c)$, and one would naively expect a small normal state PC. We find, however, that close to the critical mean field line (see Fig. 1) there is a parametrically large enhancement of the PC due to quantum fluctuations that decays slowly away from that line. The magnitude for the normal PC is $I_N \sim \frac{1}{\phi_0}\frac{D}{R^2}\frac{1}{\log g}$, where D is the diffusion coefficient and g is the dimensionless ring conductance [6,7]. Our calculations show that the PC due to pair fluctuations near φ_c is parametrically larger and at low $T \ll T_{c\varphi=0}$ given by $[T_{c\varphi=0} \equiv T_c^0]$

$$I_{\rm FL} \approx -\frac{T_c^0}{\phi_0}\frac{1}{\varphi_c}\frac{\xi}{R}\log\left(\frac{1}{\Delta\varphi}\right), \tag{1}$$

where $\Delta\varphi \equiv (\varphi - \varphi_c)/\varphi_c$ measures the distance to the critical flux φ_c. When increasing T the PC initially grows before going through a maximum at finite T, where it can considerably exceed the result of Eq. (1) [see Fig. 3]. Since

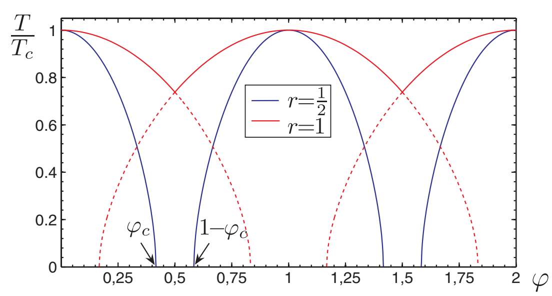

FIG. 1 (color online). MF phase diagram. $T_{c\varphi}$ separates the metallic (high T) and the superconducting (low T) phases as a function of the flux $\varphi = \phi/\phi_0$ through the ring. For sufficiently small rings with effective radius $r = R/\xi < 0.6$, MF theory predicts a full reduction of T_c for fluxes between $\varphi_c \approx 0.83r$ and $1-\varphi_c$ near $\varphi = 1/2$. The transition line reflects the condition $\mathcal{L}_{00}^{-1} = 0$; cf. Eq. (8). The dotted lines give $T_{n\varphi}$ defined below Eq. (3) for $n \in \{0, 1, 2\}$.

$r^{-1} = \xi/R$ is a number of order 1 and $\frac{D}{R^2} = \frac{8}{\pi}\frac{T_c^0}{r^2}$ for a weakly disordered superconductor, we find an enhancement factor of $\log(g)\log(1/\Delta\varphi)$.

Our results are obtained for the case when the flux acts as a pair-breaking mechanism. Other pair-breaking mechanisms, e.g., magnetic impurities or a magnetic field penetrating the ring itself, will lead to similar results. They cause a reduction of T_c to zero; the pair fluctuations, however, lead to a parametric enhancement of the PC in the normal state. Reference [8] suggests that a similar mechanism due to magnetic impurities is related to the unexpectedly large PC in noble metal rings [9,10].

A metallic state with small but finite resistance was observed experimentally in superconducting cylinders [4] with $\varphi \approx 1/2$. Further studies will be needed to clarify the relation to our findings, where a large PC is caused by pair fluctuations that are unable to condense.

II. Regime with $r > 1$.—The case $r > 1$ is suitable for the description of the experiments on persistent currents by Koshnick *et al.* [5]. Previously, this experiment has been interpreted using a one-dimensional Ginzburg-Landau (GL) theory to describe the order parameter fluctuations [11]. Following these lines, one has to resort to numerical methods [12] in order to describe the critical region close to T_c, where fluctuations proliferate.

Our key observation is that part of the rings in the experiment allow for a description using a suitable generalization [13] of the 0d Ginzburg-Landau theory. Indeed, following an expansion of the order parameter field $\psi(\vartheta)$ in terms of angular momentum modes ψ_n, a simple physical picture arises in the limit $\sqrt{g} \gg r$. Two of the modes compete with each other close to half-integer flux, while at the same time both of them strongly fluctuate in the critical regime close to T_c.

Formally, the competition arises due to the quartic term in the GL functional that induces an interaction between the modes [14] and reveals itself in the experiment mostly in the slope of the PC, the susceptibility $\chi = -\frac{\partial I}{\partial \phi}$. With this insight χ can be calculated analytically even in the critical fluctuation regime.

As an example, denoting the susceptibility at T_c and zero flux by χ_0 and at $T_{c,\varphi=1/2}$ by $\chi_{1/2}$, we find

$$\chi_{1/2}/\chi_0 \approx -2.7\sqrt{g}/r. \tag{2}$$

Experimentally, a strong enhancement of the magnetic susceptibility near $\varphi = 1/2$ compared to $\varphi \approx 0$ was observed, and Eq. (2) demonstrates that it is controlled by the parameter $\sqrt{g}/r$. If it is large, the current will rapidly change sign as a function of the flux at half-integer flux, leading to a sawtoothlike shape of i_φ. The full T dependence of $\chi_{\varphi=1/2}$ is given in Eq. (7). For the smallest rings in Ref. [5], $\sqrt{g} \approx 33r$ (see [15]).

Classical GL functional.—Now we discuss more details of our approach starting with the description of rings with only a moderate suppression of T_c (i.e., $r \gtrsim 1$).

When the superconducting coherence length $\xi(T)$ and the magnetic penetration depth $\lambda(T)$ are much larger than the ring thickness, the system is well described by a one-dimensional order parameter field ψ [16]. The partition function can be written as a weighted average over configurations of the order parameter ψ: $\mathcal{Z} = \int D\psi \exp[-\mathcal{F}/T]$. By introducing angular momentum modes as $\psi(\vartheta) = (1/\sqrt{V})\sum_n \psi_n e^{in\vartheta}$, where V is the volume of the ring, the free energy functional takes the form

$$\mathcal{F} = \sum_n a_{n\varphi}|\psi_n|^2 + \frac{b}{2V}\sum_{nmkl}\delta_{n+k,l+m}\psi_n\psi_m^*\psi_k\psi_l^*. \tag{3}$$

Here $a_{n\varphi} = \alpha T_c^0 \varepsilon_{n\varphi}$, where $\varepsilon_{n\varphi} = (T - T_{n\varphi})/T_c^0$ is the reduced temperature, and $T_{n\varphi} = T_c^0[1 - (n-\varphi)^2/r^2]$ is determined by the sign change of the coefficient $a_n(\varphi)$ and can thus loosely be interpreted as the transition temperature of mode ψ_n [17]. The mean field transition occurs at $T_{c\varphi}$ that is equal to the maximal T_n for given φ, i.e., at the point where the first mode becomes superconducting when lowering the temperature (cf. Fig. 1). The 0d Ginzburg parameter $Gi = (2b/\alpha^2 T_c V)^{1/2}$ is an estimate for the width of the critical regime in the variable ε_n. The parameter $\sqrt{g}/r \approx 1/5r^2 Gi$ has been used when stating our results. Its relevance is now easily understood. $1/r^2$ is a measure for the typical spacing between the transition temperatures T_n for different modes, since $(T_0 - T_1)/T_c^0 = (1 - 2\varphi)/r^2$. This spacing should be compared to the typical width of the non-Gaussian fluctuation region Gi. If it is large, a theory including only one or two angular momentum modes is applicable.

Persistent current.—The persistent current I is found from the free energy $F = -T\ln\mathcal{Z}$ by differentiation $I = -\partial F/\partial\phi$. The normalized current is given by

$$i = I/(T_c/\phi_0) = \sum_{n=-\infty}^{\infty}\frac{2\alpha}{r^2}(n-\varphi)\langle|\psi_n|^2\rangle. \tag{4}$$

The averaging is performed with respect to the functional $\mathcal{F}$ in Eq. (3). i_φ is periodic in the flux φ with period one.

Case $\varphi \approx 0$.—The most important contribution in the regime of non-Gaussian fluctuations close to integer fluxes comes from the angular momentum mode ψ_n with the highest transition temperature $T_{n\varphi}$. One may then approximate Eq. (3) by a single mode and calculate with $\mathcal{F}_n = a_n|\psi_n|^2 + \frac{b}{2V}|\psi_n|^4$ [18]. This is the 0d limit of the GL functional [19] where the functional integral becomes a conventional integral. Indeed, performing the integral in polar coordinates, one finds $\mathcal{Z} = (\pi\sqrt{\pi}/\alpha Gi)\times \exp(x_n^2)\mathrm{erfc}(x_n)$, where $x_n = \varepsilon_n/Gi$ [20]. Using now Eq. (4) with one mode only, we find

$$i_n = 4\Lambda(n-\varphi)f(x_n) \quad \text{for } \varphi \approx n. \tag{5}$$

Here $\Lambda \equiv 1/r^2 Gi \approx 5\sqrt{g}/r$ and $f(x) = [\exp(-x^2)/\sqrt{\pi}\mathrm{erfc}(x)] - x$ [20]. We note in passing the high degree of universality implied by this result: All PC measurements will fall on the same curve, if the PC—measured in suitable units $i = I/(T_c^0/\phi_0)$—and the reduced temperature $\varepsilon_\varphi = (T - T_{c\varphi})/T_c^0$ are scaled as $i \to i(r/\sqrt{g})$, $\varepsilon_\varphi \to \varepsilon_\varphi r\sqrt{g}$. This relation is a valuable guide in characterizing different rings in experiments.

Far above T_c, one obtains as a limiting case the Gaussian result for a single mode $i_n \approx 2(n - \varphi)/r^2\varepsilon_{n\varphi}$, that can also be obtained directly by neglecting the quartic term in the GL functional. It is known, however, that as soon as temperatures are too high, $\varepsilon_n \gg 1/r^2$, it is important to sum the contribution of all modes [21]. Far below T_c, one recovers the mean field result $i_{\rm MF} \equiv \frac{-4}{r^2 Gi^2}\varepsilon_{n\varphi}(n - \varphi)$ for the PC in the superconducting regime. The PC i_n in Eq. (5) interpolates smoothly between the Gaussian and the mean field result.

Case $\varphi \approx 1/2$.—A very interesting situation occurs at half-integer values of φ. The transition temperatures for two modes become equal, their *coupling* becomes crucial ($\varphi \approx 1/2$ for definiteness), and we approximate [13]

$$\mathcal{F} = \sum_{i=0,1} a_i|\psi_i|^2 + \frac{b}{2\mathrm{Vol}}(|\psi_0|^4 + |\psi_1|^4 + 4|\psi_0|^2|\psi_1|^2). \tag{6}$$

Calculation of the PC in the presence of the coupling requires a generalization of the approach used for the single-mode case [13,22]. In Fig. 2, we display the PC i_2 as calculated from Eq. (6) for three different temperatures: $T_{c(1/2)} < T < T_c^0$, $T = T_{c(1/2)}$, and $T < T_{c(1/2)}$. We compare it to the MF result as well as to i_{20} obtained by neglecting the coupling $|\psi_0|^2|\psi_1|^2$ in Eq. (6).

Above $T_{c\varphi}$ ($\varepsilon = -0.05$ in Fig. 2), in the region where the mean field result vanishes near half-integer flux, the PC is *purely fluctuational*. We deduce from Fig. 2 that the coupling of the modes is crucial for $\chi(1/2)$ but not for the overall shape when $T > T_{c(1/2)}$. However, just below $T_{c(1/2)}$ ($\varepsilon = -0.11$ in Fig. 2), the coupling is essential. The mean field result is not applicable as it gives an infinitely sharp jump in the PC at half-integer flux. The result without coupling of the modes, i_{20}, gives a finite slope, but it is far from the full current i_2 that includes the mode coupling. The coupling drives the current i_2 towards the mean field approximation $i_{\rm MF}$ which includes only one mode. This occurs because for a repulsive coupling the dominant mode suppresses the subdominant one.

Susceptibility.—We will now discuss in more detail the slope at half-integer flux, which is most sensitive to the coupling between the modes below and to the non-Gaussian fluctuations close to T_c. Differentiating the expression [22] for i_2, we obtain

$$\bar{\chi}_{\varphi=1/2} \equiv \chi/(T_c^0/\phi_0^2) = 4\Lambda g_1(x) - 4\Lambda^2 g_2(x), \tag{7}$$

where $x = \frac{\varepsilon}{Gi} + \frac{1}{4}\Lambda$ [23]. The dimensionless smooth functions $g_1(x) = \frac{1}{2J(x)}e^{(1/3)x^2}\mathrm{erfc}(x) - \frac{2x}{3}$ and $g_2(x) = [3/2\sqrt{\pi}J(x)]e^{-(2/3)x^2} - \frac{3x}{2J(x)}e^{(1/3)x^2}\mathrm{erfc}(x) - 1$, where $J(x) = \int_x^\infty dt e^{(1/3)t^2}\mathrm{erfc}(t)$, obey $g_1(0) \approx 0.78$ and $g_2(0) \approx$

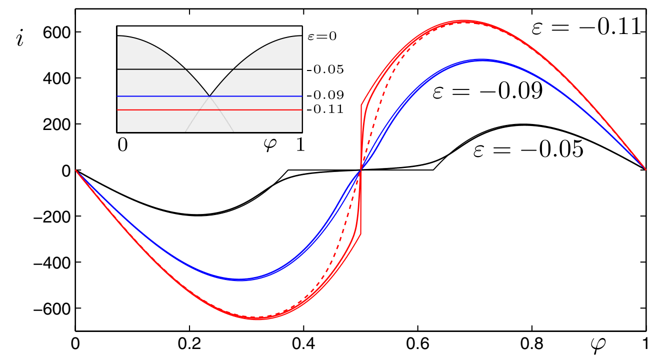

FIG. 2 (color online). The PC $i = I/(T_c^0/\phi_0)$ as a function of the flux φ. Parameters are $r = R/\xi = 1.66$, $\Lambda = 1/(r^2 Gi) \approx 5\sqrt{g}/r = 50$, and $\varepsilon = (T - T_c^0)/T_c^0$. The transition for $\varphi = 1/2$ occurs at $\varepsilon = -0.09$. Full lines: i_2 calculated with $\mathcal{F}$ of Eq. (6); it takes into account two modes and the interaction between them. We compare i_2 to two approximations, which neglect this interaction. Dotted lines: The mean field approximation $i_{\rm MF}$ [discussed before Eq. (6)]. Dashed line: i_{20} calculated with $\mathcal{F}$ of Eq. (6) *without* coupling [28]. Inset: MF phase diagram, superconducting region in gray.

0.315. For large $\Lambda = 1/r^2 Gi \approx 5\sqrt{g}/r$, one can neglect the first term in Eq. (7). Then one obtains $\bar{\chi}_{1/2} = -4\Lambda^2 g_2(x)$. For the susceptibility close to integer flux, one easily obtains $\bar{\chi}_0 = 4\Lambda f(x_0)$ from Eq. (5). Comparing to the expression for $\bar{\chi}_{1/2}$, we find Eq. (2). This is the strong enhancement of $\chi_{1/2}$ compared to χ_0 observed in the experiment [5].

Quantum critical regime.—So far, we have discussed the limit $r = R/\xi > 1$, where the suppression of T_c is small and a finite temperature phase transition occurs. We will now discuss the case where $r = R/\xi < 0.6$ and T_c is reduced to zero at a critical flux φ_c near $\varphi = 1/2$; see Fig. 1. Near the quantum critical point (QCP), it is no longer legitimate to use the classical GL functional, in

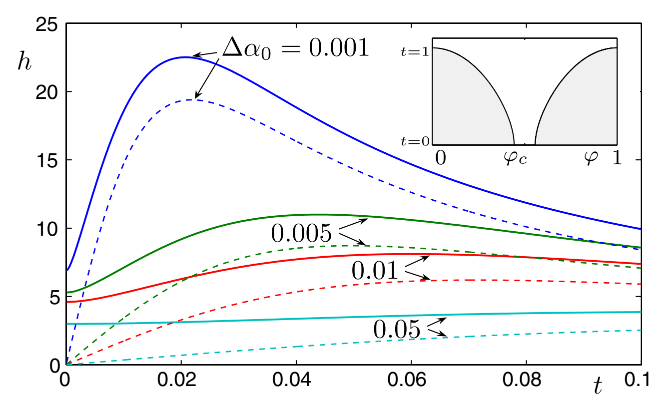

FIG. 3 (color online). The function h that determines the PC close to the QCP, $i_G \approx -\frac{2}{\gamma_E}\frac{1}{\varphi_c}h$, as a function of $t = T/T_c^0$ for given $\Delta\alpha_0 \approx 2(\varphi - \varphi_c)/\varphi_c$. $\varphi_c = \pi r/2\sqrt{2\gamma_E}$ is the critical flux at $T = 0$ and $\gamma_E \approx 1.78$. h is defined in the text. The dotted lines describe classical fluctuations. Inset: MF phase diagram, superconducting region in gray.

which only the static component of the order parameter field is considered. Instead, all Matsubara frequencies should be taken into account in the imaginary time formalism. The full fluctuation propagator is given by

$$(\nu\mathcal{L})^{-1}_{nk} = \ln\left[\frac{T}{T_c^0}\right] + \psi\left[\frac{1}{2} + \frac{\alpha_n + |\Omega_k|/2}{2\pi T}\right] - \psi\left[\frac{1}{2}\right], \quad (8)$$

where $\alpha_n(\varphi) = \frac{1}{2}\frac{D}{R^2}(n-\varphi)^2$ and $\Omega_k = 2\pi kT$ are bosonic Matsubara frequencies [24,25]. Following the standard approach, we first find the critical line $\varphi(T)$ in the temperature-flux plane by equating $\mathcal{L}_{00}^{-1} = 0$. For the QCP at $T = 0$, one obtains the critical flux $\varphi_c = \pi r/(2\sqrt{2\gamma_E})$, where $\gamma_E \approx 1.78$ [3]. Because of the flux periodicity of the phase diagram, the QCP can be observed in the ring geometry only if $\varphi_c < 1/2$ which implies $r < \sqrt{2\gamma_E}/\pi \approx 0.6$. Notice that this critical value of $r = R/\xi$ is 20% larger than a naive application of the quadratic approximation valid for $r \gg 1$ would suggest.

Restricting ourselves to the interval $\varphi \in (0, 0.5)$, we find that near the QCP it is sufficient to consider the $n = 0$ mode. In the Gaussian regime we obtain (cf. Fig. 1) the following fluctuation contribution to the PC: $i_G = -\frac{2\nu\varphi}{T_c^0\varphi^2(T)} T\sum_k \mathcal{L}_{0k}$ [26]. Expanding $\mathcal{L}_0^{-1}$ in small $\Delta\alpha = \{\alpha_0(\varphi) - \alpha[\varphi(T)]\}/\alpha_0(\varphi_c)$, we find with logarithmic accuracy $i_G = -\frac{16\varphi}{\pi^2 r^2} h(\Delta\alpha, t) \xrightarrow{\varphi\to\varphi_c} -\frac{2}{\gamma_E}\frac{1}{\varphi_c} h(\Delta\alpha, t)$, where $h(\Delta\alpha, t) = \ln\frac{s}{\Delta\alpha} + \frac{1}{2s} - \psi(1+s)$, $s = \frac{\Delta\alpha}{2\gamma_E t}$, and $t = T/T_c^0$.

A few remarks are in order concerning this result. The second term in the expression for h is the classical $\Omega = 0$ contribution to the sum. The upper cutoff for the frequency summation has been chosen as $\bar{\Omega} = 2\alpha_0[\varphi(T)]$ [27]. The function h has the asymptotic form $h \approx \gamma_E t/\Delta\alpha + \ln(1/2\gamma_E t)$ for $\Delta\alpha \ll t \ll 1$ and $h \approx \ln(1/\Delta\alpha)$ for $t \ll \Delta\alpha \ll 1$. It is important that $\Delta\alpha$ is T-dependent, and in order to reveal the full T dependence of i_G one should first find the transition line $\alpha_0[\varphi(T)]$. $h(T)$ is displayed in Fig. 3. The maximum of h at finite T is a result of two competing mechanisms. As T grows from zero, thermal fluctuations become stronger. At the same time, the distance to the critical line becomes larger for fixed φ, which eventually leads to a decrease of i_G.

Conclusion.—In conclusion, we showed that, on the normal side of the flux-tuned superconductor normal-metal transition in small rings, the fluctuation PC can be very large compared to the normal case and decays only logarithmically away from the critical point. For larger rings as studied in recent experiments, we obtained analytical predictions for the strong fluctuation region.

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[27] The divergence is cured by the contribution of all nonsingular modes with $n \neq 0$.
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